High energy QCD & the Colour Glass Condensate

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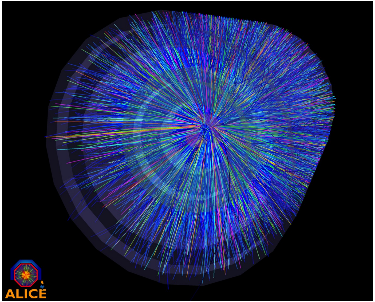
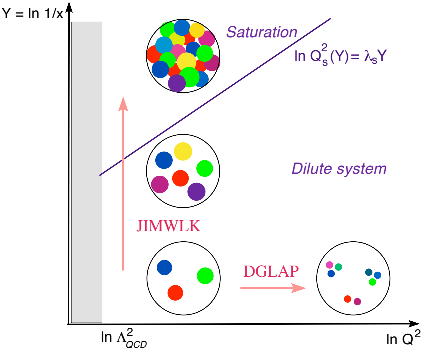
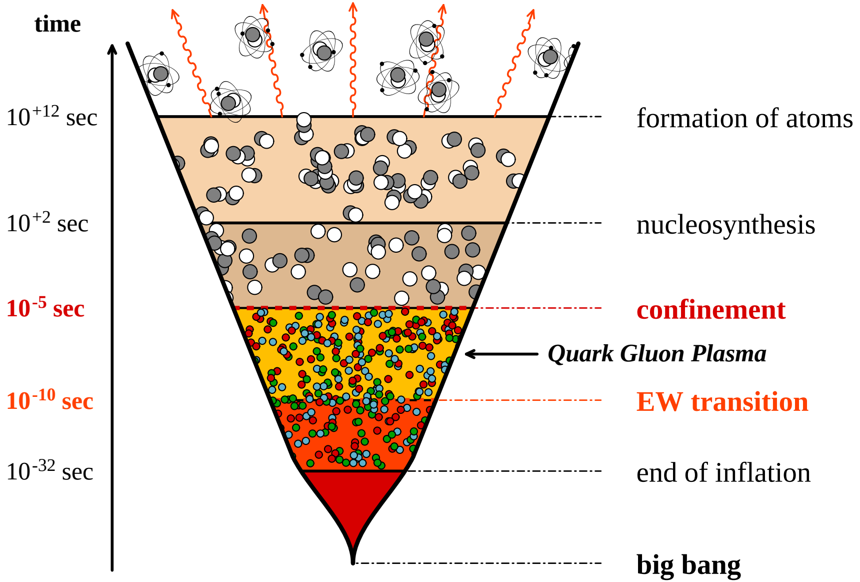




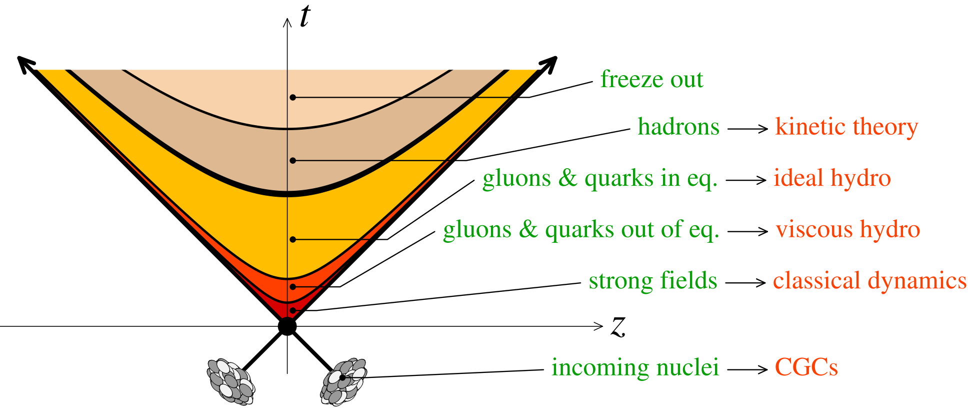
The Big Bang



time
10^{+12} sec
10^{+2} sec
10^{-5} sec
10^{-10} sec
10^{-32} sec
formation of atoms
nucleosynthesis
confinement
Quark Gluon Plasma
EW transition
end of inflation
big bang

The Little Bang

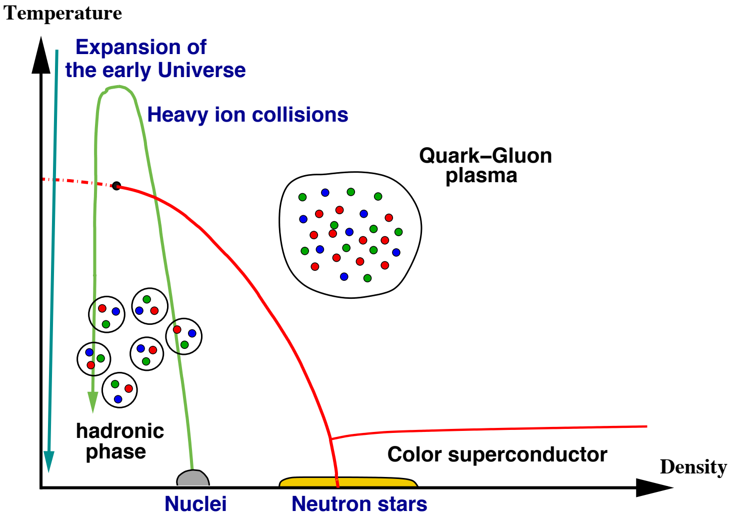
- A space–time picture of a heavy ion collision (HIC)



- 'Initial singularity' : the collision between the two incoming nuclei
- The QGP is re-created in the intermediate stages

Phase–diagram for QCD

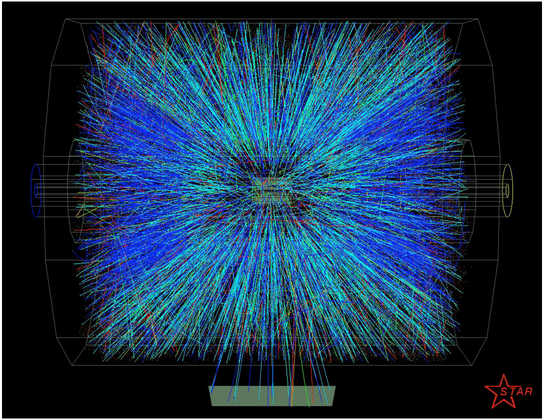
- ... as explored by the expansion of the Early Universe ...



- ... and in the ultrarelativistic heavy ion collisions.

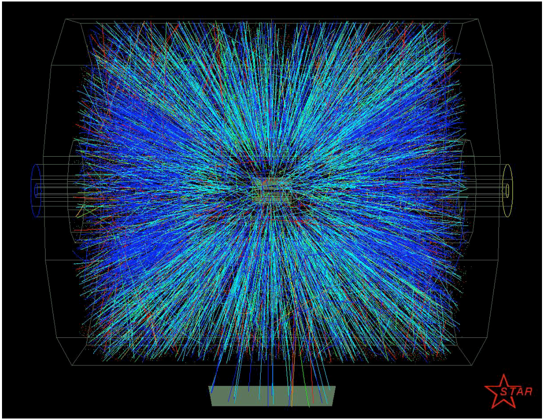
Heavy Ion Collisions @ RHIC & the LHC

Au+Au collisions at RHIC



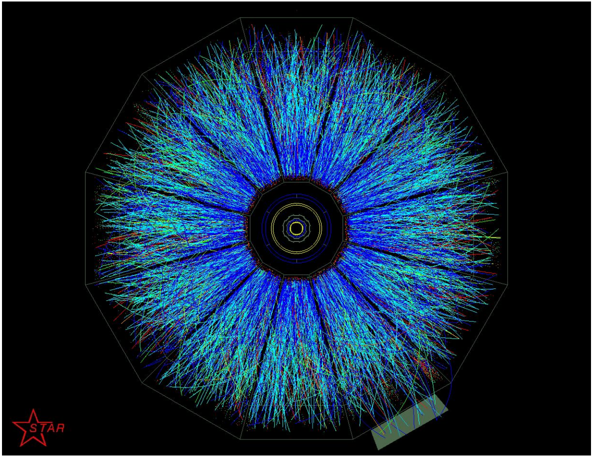
- Au+Au collision at STAR: longitudinal projection
- $\sim$ 7000 produced particles streaming into the detector
- Collision energy (COM frame) : $\sqrt{s} = 200$ GeV/nucleon

Au+Au collisions at RHIC



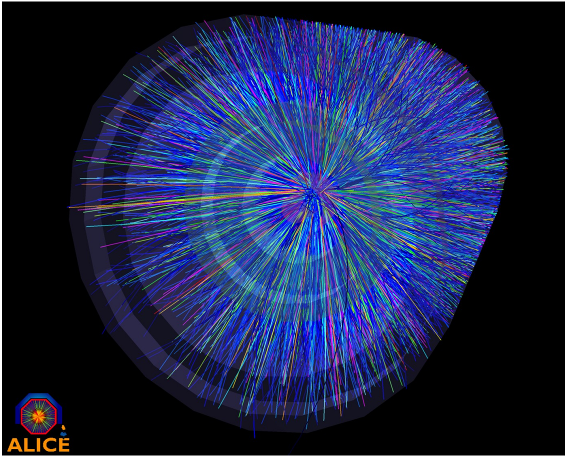
- Au+Au collision at STAR: along the beam axis
- $\sim$ 7000 produced particles streaming into the detector
- Collision energy (COM frame) : $\sqrt{s} = 200$ GeV/nucleon

Au+Au collisions at RHIC



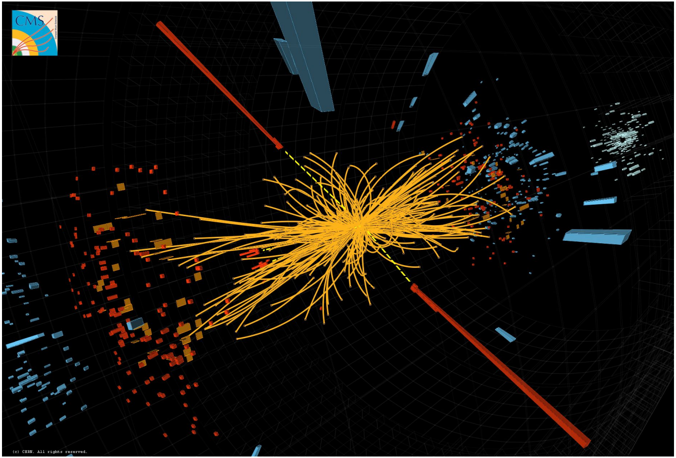
- Au+Au collision at STAR: transverse projection

Pb+Pb collisions at the LHC: ALICE



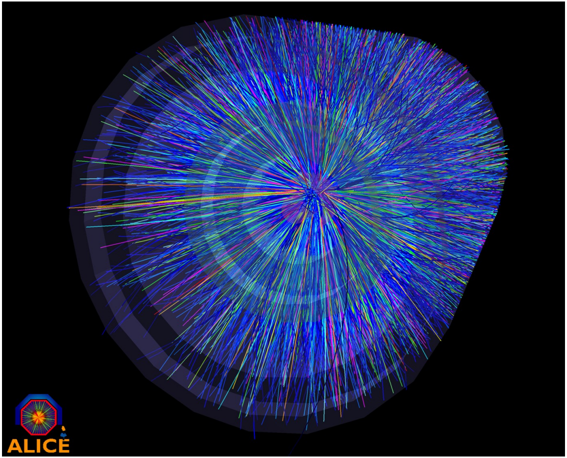
- Pb+Pb collision at ALICE: $\sqrt{s} = 2760$ GeV/nucleon
- $\gtrsim$ 20,000 hadrons in the detectors
- Is that much ?

p+p collisions at the LHC: CMS



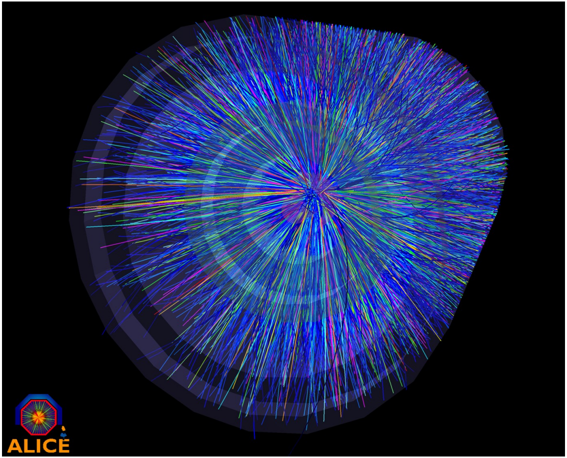
- p+p collision at 7 TeV: candidate event for $H \to \gamma\gamma$
- Less than 50 tracks/hadrons in the final state

Pb+Pb collisions at the LHC: ALICE



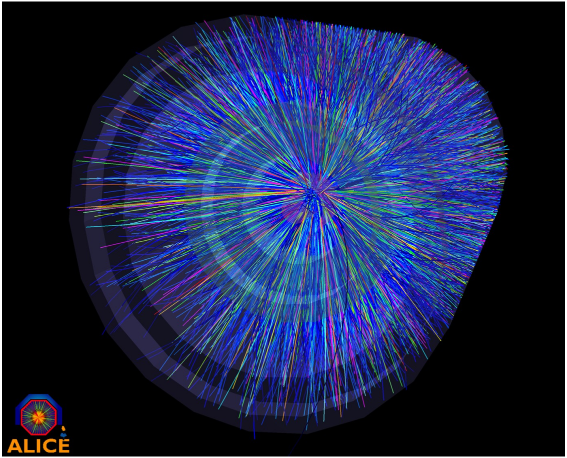
- Where are all these (> 10000) hadrons coming from ?
- How to trace back their history ?
- How to understand that from first principles (QCD) ?

Pb+Pb collisions at the LHC: ALICE



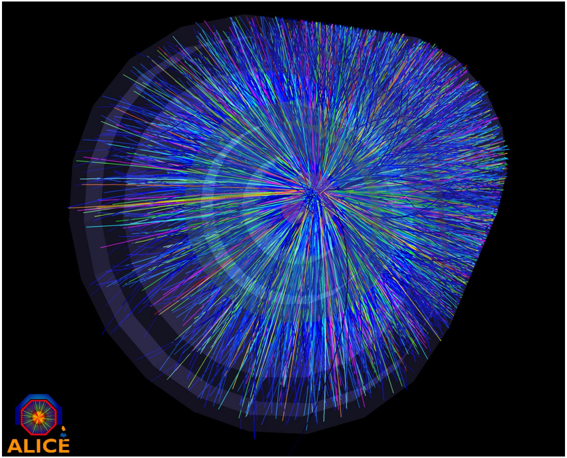
- Partons which have been liberated by the collision.
- How to trace back their history ?
- How to understand that from first principles (QCD) ?

Pb+Pb collisions at the LHC: ALICE



- Partons which have been liberated by the collision.
- They leave imprints on the hadron distribution in the final state.
- How to understand that from first principles (QCD) ?

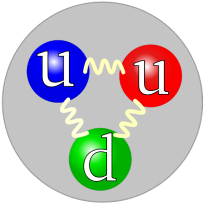
Pb+Pb collisions at the LHC: ALICE

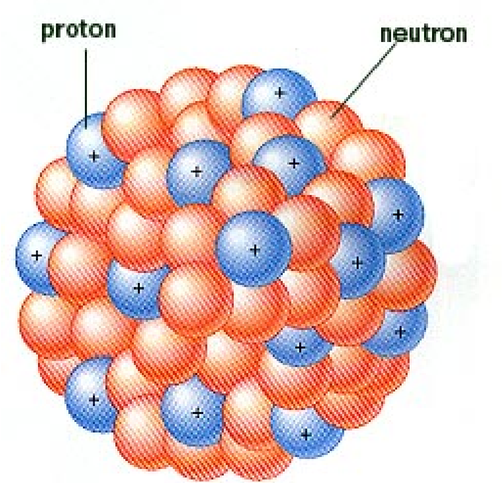


- Partons which have been liberated by the collision.
- They leave imprints on the hadron distribution in the final state.
- Build effective theories for the relevant degrees of freedom.

QCD matter: from hadrons ...

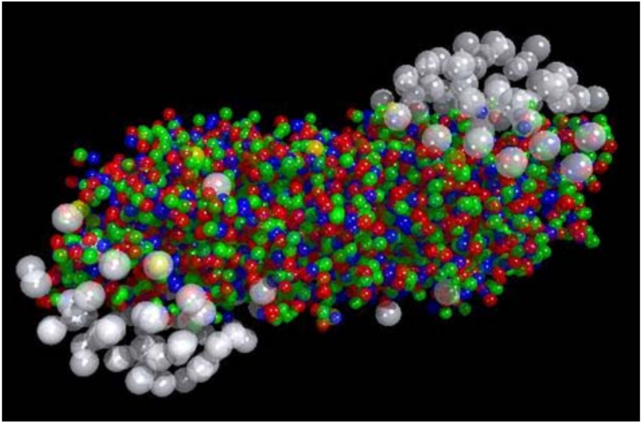
Quark composition of a pion





- At low energies, QCD matter exists only in the form of hadrons (mesons, baryons, nuclei) ... as a consequence of confinement

QCD matter: ... to partons

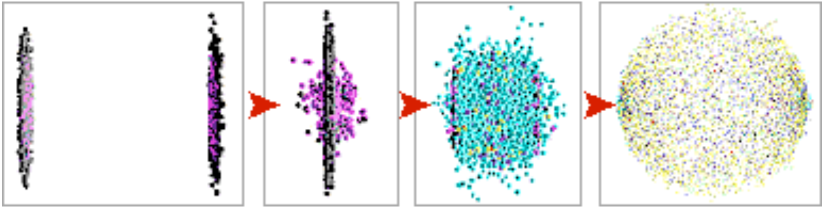


- At high energies, the relevant d.o.f. are partonic (quarks & gluons)

  ▷ interactions occur over distances much shorter than the confinement scale

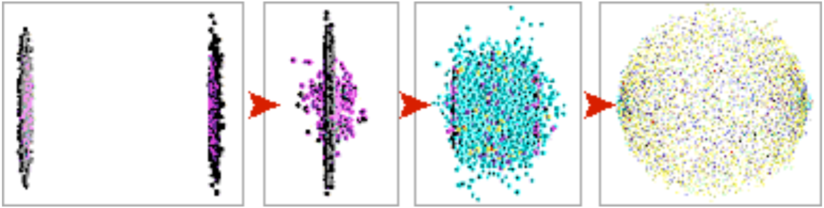
- The HIC's give us access to dense forms of partonic matter

New forms of QCD matter produced in HIC



- Prior to the collision: 2 Lorentz–contracted nuclei ('pancakes')
  - 'Color Glass Condensate' : highly coherent form of gluonic matter
- Right after the collision: non–equilibrium partonic matter
  - 'Glasma' : color fields break into partons
- At later stages ($\Delta t \gtrsim 1$ fm/c) : incomplete equilibration
  - 'Quark–Gluon Plasma' (QGP)
- Final stage ($\Delta t \gtrsim 10$ fm/c) : hadrons
  - 'final event', or 'particle production'

New forms of QCD matter produced in HIC



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  - 'Quark–Gluon Plasma' (QGP)
- My focus here: the partonic phases at early and intermediate stages

Outline

- The wavefunctions of the incoming hadrons:

  Color Glass Condensate

- Particle production at early stages :

  proton-proton (pp), proton-nucleus (pA), nucleus-nucleus (AA)

- AA collisions : Glasma & thermalization

- Flow and hydrodynamics

- Thermodynamics of the Quark Gluon Plasma

- Hard probes of the QGP: jet quenching

▷ Main emphasis: What do heavy ion collisions teach us about QCD

- For a general and rather elementary introduction and for more references (albeit a bit outdated), you may have a look at this review paper:

arXiv.org > hep-ph > arXiv:1205.0579 Search or Artic

High Energy Physics – Phenomenology

QCD in heavy ion collisions

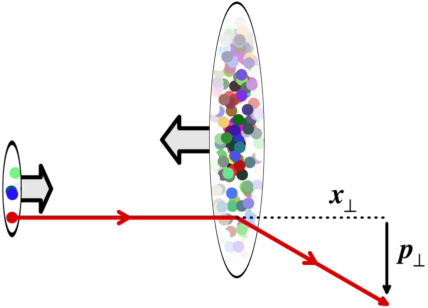
Edmond Iancu

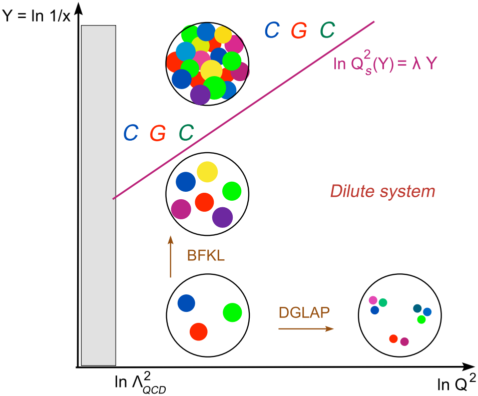
(Submitted on 2 May 2012)

These lectures provide a modern introduction to selected topics in the physics of ultrarelativistic heavy ion collisions which shed light on the fundamental theory of strong interactions, the Quantum Chromodynamics. The emphasis is on the partonic forms of QCD matter which exist in the early and intermediate stages of a collision -- the colour glass condensate, the glasma, and the quark-gluon plasma -- and on the effective theories that are used for their description. These theories provide qualitative and even quantitative insight into a wealth of remarkable phenomena observed in nucleus-nucleus or deuteron-nucleus collisions at RHIC and/or the LHC, like the suppression of particle production and of azimuthal correlations at forward rapidities, the energy and centrality dependence of the multiplicities, the ridge effect, the limiting fragmentation, the jet quenching, or the dijet asymmetry.

Comments:	Based on lectures presented at the 2011 European School of High-Energy Physics, 7-20 September 2011, Cheile Gradistei, Romania. 73 pages, many figures
Subjects:	**High Energy Physics – Phenomenology (hep-ph)**; High Energy Physics – Experiment (hep-ex); Nuclear Theory (nucl-th)
Cite as:	arXiv:1205.0579 **[hep-ph]** (or arXiv:1205.0579v1 **[hep-ph]** for this version)

High-energy QCD and the CGC

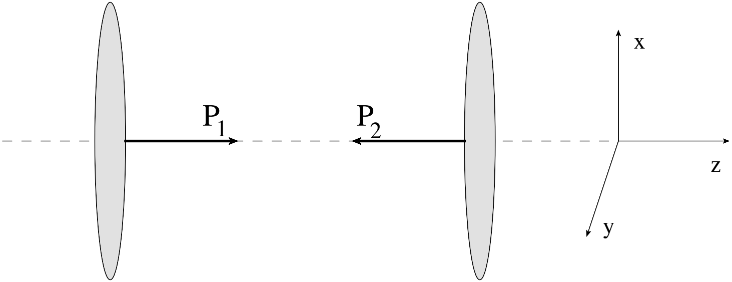




More specific references ...

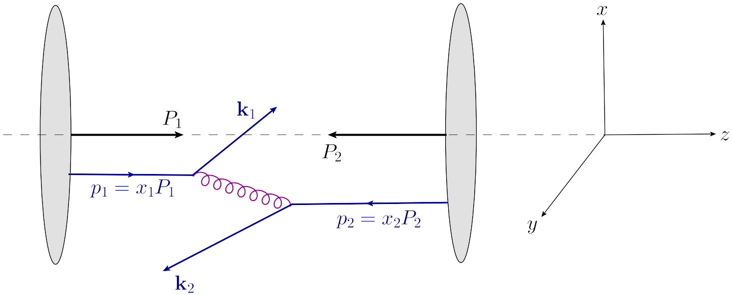
- ... on High-energy QCD and the Color Glass Condensate
- Whenever possible, I will refer to the arXiv number
- A book: *Quantum chromodynamics at high energy*, by Yuri V. Kovchegov and Eugene Levin, 2012, 349 pp. (Cambridge Univ Press)
- A few review papers or lecture notes (not exhaustive):
  - *The Colour Glass Condensate: An Introduction*, by E. Iancu, A. Leonidov, and L. McLerran, arXiv:hep-ph/0202270
  - *The Color Glass Condensate and High Energy Scattering in QCD*, by E. Iancu and R. Venugopalan, arXiv:hep-ph/0303204
  - *High energy scattering in Quantum Chromodynamics*, by F. Gelis, T. Lappi, and R. Venugopalan, arXiv:0708.0047 [hep-ph]
  - *The Color Glass Condensate*, by F. Gelis, E. Iancu, J. Jalilian-Marian, and R. Venugopalan, arXiv:1002.0333 [hep-ph]
  - *Gluon saturation and initial conditions for relativistic heavy ion collisions*, J. L. Albacete and C. Marquet, arXiv:1401.4866 [hep-ph].

A hadron-hadron collision



- pp or nucleon–nucleon (NN) pair from a pA or AA collision
- z : longitudinal (or 'beam') axis; $\boldsymbol{x}_\perp = (x, y)$: transverse plane
- Center-of-mass frame : $P_1^\mu = (E, 0, 0, E)$, $P_2^\mu = (E, 0, 0, -E)$
  - high energy: particle masses are negligible: $E = \sqrt{P_z^2 + M^2} \simeq |P_z|$
  - huge boost factor $\gamma = E/M \sim 1000$ at the LHC: Lorentz contraction
- Center-of-mass energy squared : $s = (P_1 + P_2)^2 = 2P_1 \cdot P_2 = 4E^2$

A partonic subcollision

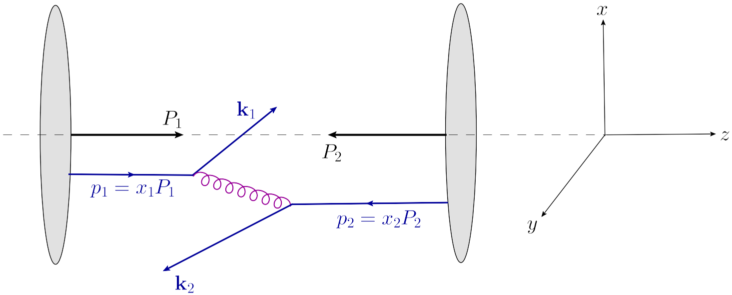


- High energy interactions truly proceed at partonic level
  - quarks and gluons from the wavefunctions of the incoming hadrons
- To lowest order in perturbative QCD (i.e. $\mathcal{O}(\alpha_s^2)$): a $2 \to 2$ subcollision
  - e.g. $q(p_1) + q(p_2) \;\rightarrow\; q(k_1) + q(k_2)$
- Initial partons are assumed to be collinear with the incoming hadrons

$$p_1^\mu = (x_1 E,\, \mathbf{0}_\perp,\, x_1 E)\,, \qquad p_2^\mu = (x_2 E,\, \mathbf{0}_\perp,\, -x_2 E)$$

- Longitudinal momentum fractions: $x = |p_z|/E$

A partonic subcollision

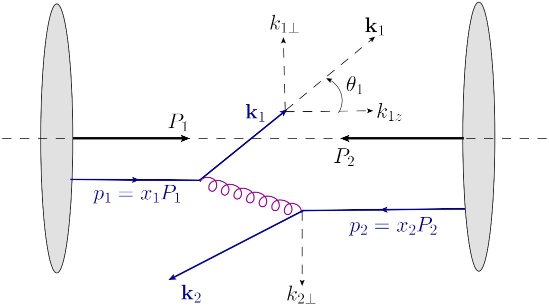


- High energy interactions truly proceed at partonic level
  - quarks and gluons from the wavefunctions of the incoming hadrons
- To lowest order in perturbative QCD (i.e. $\mathcal{O}(\alpha_s^2)$): a $2 \to 2$ subcollision
- Transverse momenta $\boldsymbol{p}_\perp = (p_x, p_y)$ are assumed to be negligible:
  - "the only source for intrinsic $p_\perp$ is confinement"

$$p_\perp \sim \Lambda_{\rm QCD} \sim 250\ \text{GeV} \;\ll\; |p_z| = xE$$

  - "collinear factorization"

$2 \to 2$ kinematics



- Recall:
  - we are working in the COM frame of the nucleon-nucleon pair
  - the initial partons have no transverse momenta
- Transverse momentum conservation: $\boldsymbol{k}_{1\perp} + \boldsymbol{k}_{2\perp} = 0 \;\Rightarrow$ "back-to-back"
- Longitudinal momentum conservation: $x_1 E - x_2 E = k_{1z} + k_{2z}$
- Energy conservation: $x_1 E + x_2 E = |\mathbf{k}_1| + |\mathbf{k}_2|$

Rapidities

- The longitudinal kinematics is conveniently dealt with by using rapidities
- Consider an on-shell particle: $p^\mu = (E, \boldsymbol{p}_\perp, p_z)$ with $E = \sqrt{m^2 + p_\perp^2 + p_z^2}$

its rapidity: $$y \equiv \frac{1}{2} \ln \frac{E + p_z}{E - p_z}$$

- Positive for a 'right-mover' $(p_z > 0)$ & negative for a 'left-mover' $(p_z < 0)$

$$E = m_\perp \cosh y, \quad p_z = m_\perp \sinh y\,, \quad \text{with} \quad m_\perp \equiv \sqrt{m^2 + p_\perp^2}$$

- y transforms via a shift under a Lorentz boost along the collision axis

$$E \to \gamma(E + \beta p_z), \quad p_z \to \gamma(p_z + \beta E) \implies y \to y + \frac{1}{2} \ln \frac{1+\beta}{1-\beta}$$

  - β: boost velocity; $\gamma \equiv 1/\sqrt{1-\beta^2}$: Lorentz boost factor
  - rapidity differences $\Delta y_{ij} = y_i - y_j$ are boost invariant

Rapidities

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- Positive for a 'right-mover' $(p_z > 0)$ & negative for a 'left-mover' $(p_z < 0)$

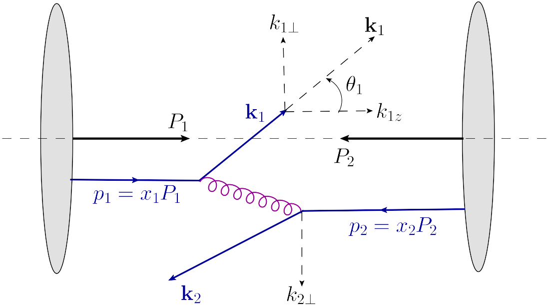
$$E = m_\perp \cosh y, \quad p_z = m_\perp \sinh y\,, \quad \text{with} \quad m_\perp \equiv \sqrt{m^2 + p_\perp^2}$$

- In the experiments, it is easier to measure angles $\Longrightarrow$ "pseudo-rapidities"

$$\eta \equiv \frac{1}{2} \ln \frac{p + p_z}{p - p_z} = -\ln \tan \frac{\theta}{2}\,, \quad \cos\theta = \frac{p_z}{p}$$

- $p \equiv |\vec{\mathbf{p}}| = \sqrt{p_\perp^2 + p_z^2} \quad \Longrightarrow \quad y = \eta$ for massless particles
- In these lectures, all particles are massless !

$2 \to 2$ kinematics revisited

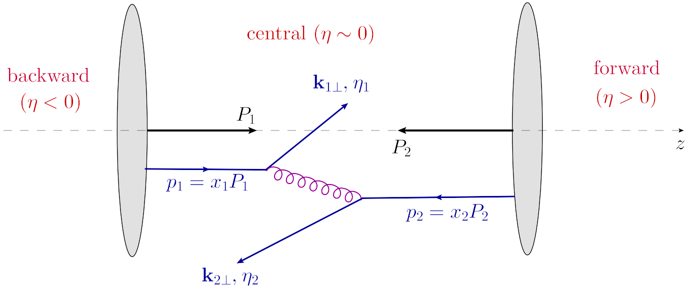


- Energy conservation: $x_1 E + x_2 E = |\mathbf{k}_1| + |\mathbf{k}_2| = k_{1\perp} \cosh \eta_1 + k_{2\perp} \cosh \eta_2$
- Longitudinal momentum: $x_1 E - x_2 E = k_{1\perp} \sinh \eta_1 + k_{2\perp} \sinh \eta_2$

$$x_1 \;=\; \frac{k_{1\perp}}{\sqrt{s}}\, \mathrm{e}^{\eta_1} + \frac{k_{2\perp}}{\sqrt{s}}\, \mathrm{e}^{\eta_2}\,, \qquad x_2 \;=\; \frac{k_{1\perp}}{\sqrt{s}}\, \mathrm{e}^{-\eta_1} + \frac{k_{2\perp}}{\sqrt{s}}\, \mathrm{e}^{-\eta_2}$$

- Particle production probes the wave functions of the incoming hadrons (their parton distributions in x)

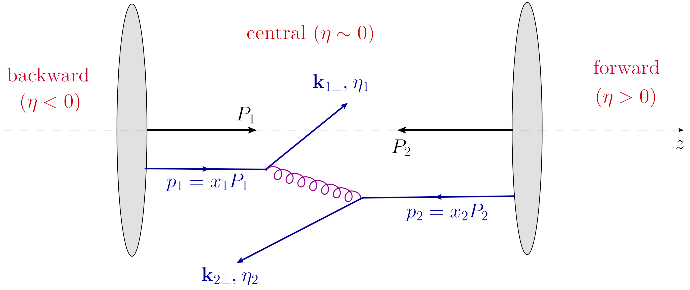
Particle production



$$x_1 = \frac{k_{1\perp}}{\sqrt{s}}\, \mathrm{e}^{\eta_1} + \frac{k_{2\perp}}{\sqrt{s}}\, \mathrm{e}^{\eta_2}\,, \qquad x_2 = \frac{k_{1\perp}}{\sqrt{s}}\, \mathrm{e}^{-\eta_1} + \frac{k_{2\perp}}{\sqrt{s}}\, \mathrm{e}^{-\eta_2}$$

- High-energy regime: $k_\perp/\sqrt{s} \ll 1 \longleftrightarrow$ small-x partons: $x \ll 1$
- "Central rapidities": $\eta \sim 0 \iff \theta \sim \pi/2 \iff |k_z| \ll k_\perp \simeq k$
- "Forward/backward rapidities": $\theta \sim 0$ or $\pi \iff |k_z| \approx k \gg k_\perp$
- Forward production : η_1, η_2 positive and large $\Longrightarrow x_2 \ll x_1$ (asymmetry)

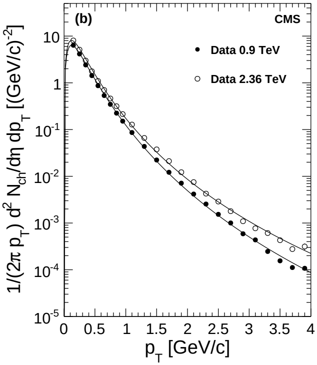
Collinear factorization *(see Markus' lectures for more !)*

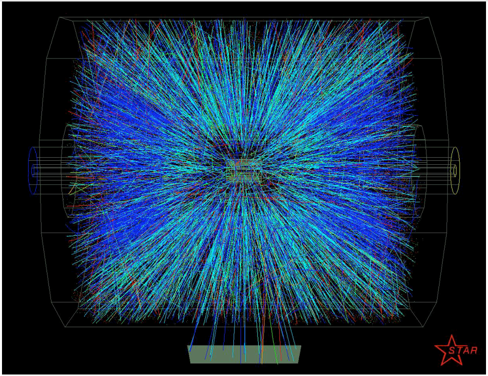


$$\frac{\mathrm{d}\sigma}{\mathrm{d}^2k_{1\perp}\mathrm{d}^2k_{2\perp}\mathrm{d}\eta_1\mathrm{d}\eta_2} = \sum_{ij} x_1 f_i(x_1,\mu^2)\, x_2 f_j(x_2,\mu^2)\, \delta^{(2)}(\boldsymbol{k}_{1\perp} + \boldsymbol{k}_{2\perp}) \frac{\mathrm{d}\hat{\sigma}_{ij}}{\mathrm{d}k_\perp^2}$$

- μ^2 : factorization scale (of order $k_\perp^2 \gg \Lambda_{\rm QCD}^2$)
- Leading-order pQCD: $\frac{\mathrm{d}\hat{\sigma}_{ij}}{\mathrm{d}k_\perp^2} \sim \frac{\alpha_s^2}{k_\perp^4} \Longrightarrow$ favors "soft" (low $k_\perp$) particles
- This eventually fails when decreasing x at fixed and moderate values of $k_\perp^2$

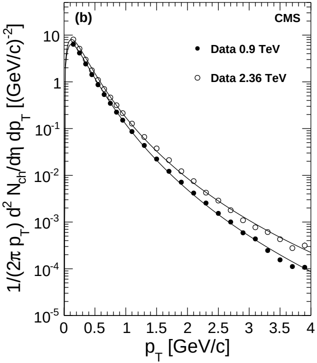
Multiplicity in pp, pA, AA : $\mathrm{d}N/\mathrm{d}\eta$

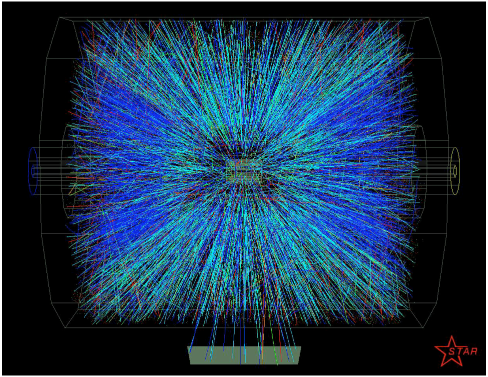




- 99% of the total multiplicity lies below $p_\perp = 2$ GeV
- $x \sim 10^{-2}$ at RHIC ($\sqrt{s} = 200$ GeV & $\eta = 0$)
- $x \sim 4 \times 10^{-4}$ at the LHC ($\sqrt{s} = 5$ TeV & $\eta = 0$)
- $x_2 \sim 10^{-5}$ at the LHC & forward rapidity ($\sqrt{s} = 5$ TeV & $\eta = 4$)

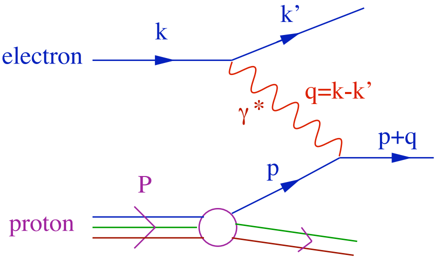
Multiplicity in pp, pA, AA : $\mathrm{d}N/\mathrm{d}\eta$



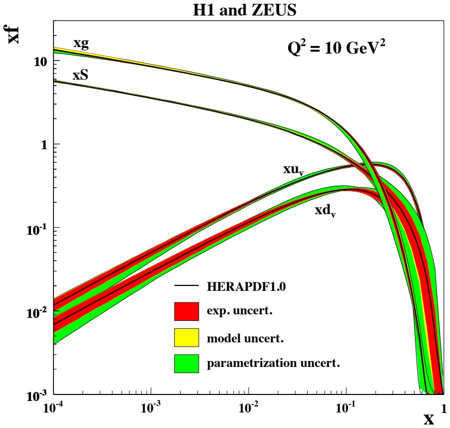


- The bulk of particle production is controlled by partons at small $x \ll 1$
- Where do all these partons come ?!

  "A nucleon is built with 3 valence quarks, each one carrying $x \sim 1/3$"
- Need to better understand the parton structure of a hadron

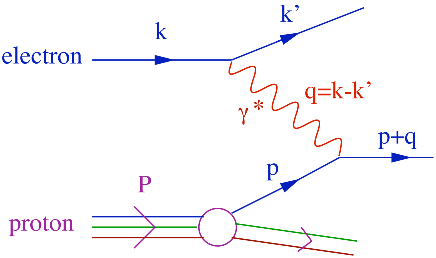


$$Q^2 \equiv -q^\mu q_\mu > 0\,, \quad x_{\rm Bj} \equiv \frac{Q^2}{2P\cdot q}$$

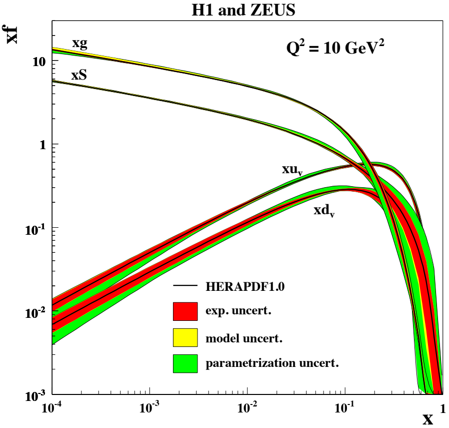


- Parton distribution functions: $xq(x, Q^2)$, $xG(x, Q^2)$

  $\triangleright$ number of partons (quark, gluons) with transverse size $\Delta x_\perp \sim 1/Q$ and longitudinal momentum fraction $x \sim x_{\rm Bj}$

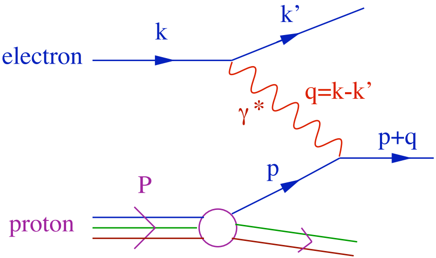


$$Q^2 \equiv -q^\mu q_\mu > 0\,, \quad x_{\rm Bj} \equiv \frac{Q^2}{2P\cdot q}$$

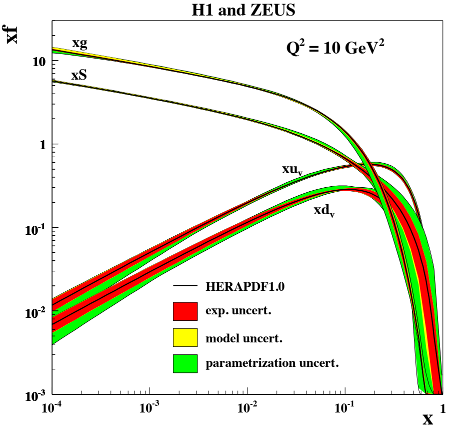


- Parton picture: a nearly on-shell, massless quark collinear with the proton absorbs the virtual photon and emerges as a free, on-shell, quark

$$0 = (p+q)^2 = p^2 + q^2 + 2p\cdot q = -Q^2 + 2xP\cdot q \Longrightarrow x = \frac{Q^2}{2P\cdot q} = x_{\rm Bj}$$



$$Q^2 \equiv -q^\mu q_\mu > 0\,, \quad x_{\rm Bj} \equiv \frac{Q^2}{2P\cdot q} \simeq \frac{Q^2}{s}$$

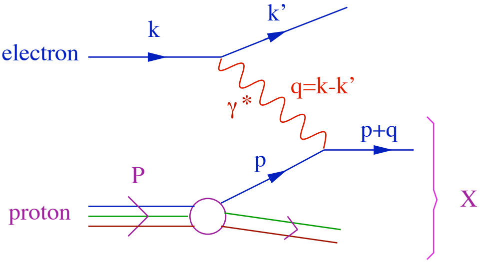


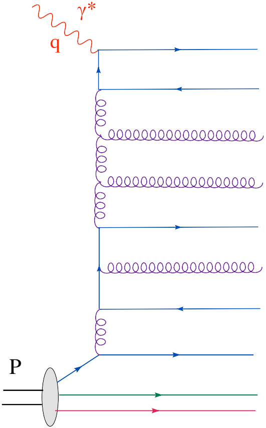
- s = COM energy squared of the proton+virtual-photon collision

$$s = (P+q)^2 = M^2 - Q^2 + 2P\cdot q \simeq 2P\cdot q \quad \text{when} \quad s \gg Q^2 > M^2$$

- DIS at high energy is probing parton distributions at small $x \ll 1$

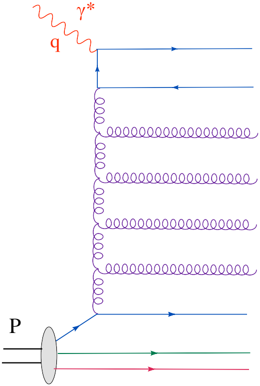
Parton evolution in QCD

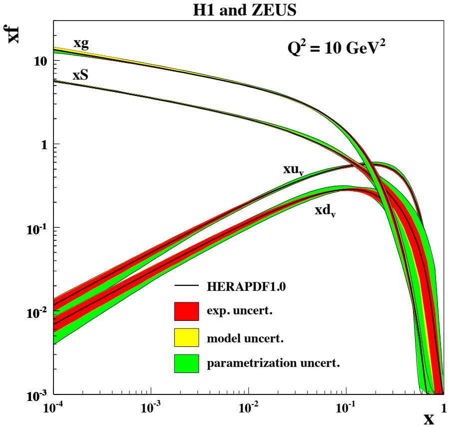




- The virtual photon γ^* couples to the (anti)quarks inside the proton
- Gluons are measured indirectly, via their effect on quark distribution
- Quantum evolution : change in the partonic content when changing the resolution scales x and Q^2, due to additional radiation

The small–x partons are mostly gluons

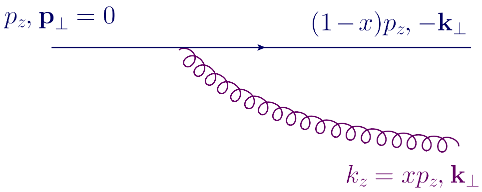




- For $x \leq 0.01$ the hadron wavefunction contains mostly gluons !
- The gluon distribution is rapidly amplified by the quantum evolution with decreasing x (or increasing energy s at fixed Q^2)

Bremsstrahlung

- A quark — say, a valence quark from a proton — emits a gluon with longitudinal momentum fraction $x \le 1$, and transverse momentum $k_\perp$



$$\mathrm{d}\mathcal{P} \simeq \frac{\alpha_s}{2\pi} \frac{\mathrm{d}k_\perp^2}{k_\perp^2} P_{g\leftarrow q}(x)\mathrm{d}x$$

$$P_{g\leftarrow q}(x) \equiv C_F \frac{1+(1-x)^2}{x}$$

- Logarithmic enhancement for large-$k_\perp$ emissions ($p^2 \sim \Lambda^2 < k_\perp^2 < Q^2$):

$$\int_{\Lambda^2}^{Q^2} \frac{\mathrm{d}k_\perp^2}{k_\perp^2} = \ln \frac{Q^2}{\Lambda^2}$$

- ... and also for soft/low-energy ($x \to 0$) gluons: $P_{g\leftarrow q}(x) \simeq 2C_F/x$

$$\int_{x_0}^{1} \frac{\mathrm{d}x}{x} = \ln \frac{1}{x_0} \equiv Y_0$$

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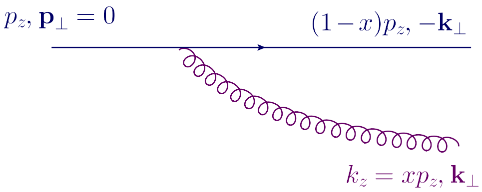

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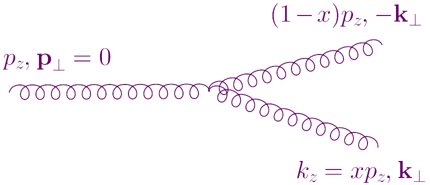
- Emissions of soft quarks are not enhanced: $\xi \equiv 1 - x \ll 1$

$$P_{q\leftarrow q}(\xi) = P_{g\leftarrow q}(x = 1-\xi) = C_F \frac{1+\xi^2}{1-\xi} \to C_F$$

Gluon splitting

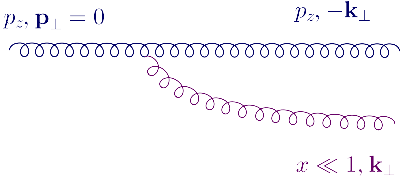
- Gluon splitting into two gluons:



$$\mathrm{d}\mathcal{P} \simeq \frac{\alpha_s}{2\pi}\,\frac{\mathrm{d}k_\perp^2}{k_\perp^2}\,P_{g\leftarrow g}(x)\mathrm{d}x$$

$$P_{g\leftarrow g}(x) \equiv 2N_c\,\frac{[1-x(1-x)]^2}{x(1-x)}$$

- Soft gluon emission: $x \ll 1 \Longrightarrow$ "eikonal approximation"



$$\mathrm{d}\mathcal{P} \simeq \frac{\alpha_s N_c}{\pi}\,\frac{\mathrm{d}k_\perp^2}{k_\perp^2}\,\frac{\mathrm{d}x}{x}$$

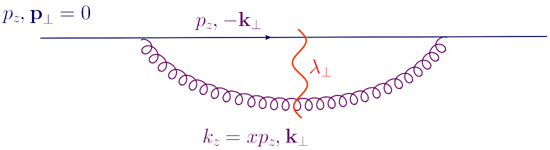
- The transverse position of the parent parton is not changing during the lifetime of the fluctuation

The fluctuation lifetime

- An on-shell parton cannot decay into a pair of on-shell partons

  ▷ the gluon is eventually reabsorbed: "virtual fluctuation"

- The maximal transverse separation $\sim$ transverse wavelength



$$\Delta x_\perp \sim \frac{k_\perp}{k_z}\Delta t \lesssim \lambda_\perp \sim \frac{2}{k_\perp}$$

$$\Delta t \simeq \frac{2k_z}{k_\perp^2}$$

- It is the transverse velocity $v_\perp = k_\perp/k_z$ of the soft gluon which matters

- The transverse velocity $V_\perp = k_\perp/p_z = xv_\perp$ acquired by the energetic parent parton is negligible: $\Delta X_\perp \sim V_\perp \Delta t \simeq x\lambda_\perp \ll \lambda_\perp$

- N.B. The same estimate for Δt follows from the uncertainty principle

$$\frac{1}{\Delta t} = \Delta E \equiv \sqrt{(xp_z)^2 + k_\perp^2} + \sqrt{((1-x)p_z)^2 + k_\perp^2} - p_z \simeq \frac{k_\perp^2}{2x(1-x)p_z}$$

The fluctuation lifetime

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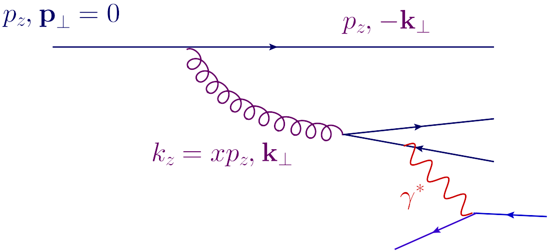


$$\Delta t \simeq \frac{2xp_z}{k_\perp^2}$$

$$\Delta t_{coll} \simeq \frac{1}{q_0}$$

$$x \simeq \frac{Q^2}{s} \simeq \frac{Q^2}{2p_z q_0}$$

- In DIS, the virtual photon "sees" only those fluctuations which live long enough: longer than the collision time

$$\Delta t \gtrsim \Delta t_{coll} \implies k_\perp^2 \lesssim Q^2$$

The gluon distribution of a single quark

- To leading order in α_s: single gluon emission by the quark $\Longrightarrow$

$$\frac{\mathrm{d}N_{\mathrm{gluon}}}{\mathrm{d}x\mathrm{d}^2k_\perp} = \frac{\mathrm{d}\mathcal{P}_{\mathrm{Brem}}}{\mathrm{d}x\mathrm{d}^2k_\perp}$$

$\triangleright$ "unintegrated gluon distribution"

- The gluon distribution $xG(x, Q^2)$: # of gluons with a given energy fraction x and any transverse momentum $k_\perp \lesssim Q$

$$xG^{(0)}(x, Q^2) = \int^Q \mathrm{d}^2\boldsymbol{k}\, x\frac{\mathrm{d}N_{\mathrm{gluon}}}{\mathrm{d}x\mathrm{d}^2k_\perp} = \frac{\alpha_s C_F}{\pi}\int_{\Lambda^2}^{Q^2}\frac{\mathrm{d}k_\perp^2}{k_\perp^2} = \frac{\alpha_s C_F}{\pi}\ln\frac{Q^2}{\Lambda^2}$$

$\triangleright$ logarithmic sensitivity to the hard resolution scale Q^2

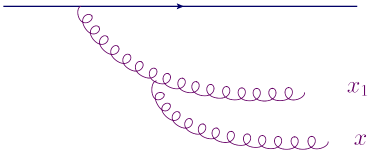
$\triangleright$ logarithmic sensitivity to the confinement scale Λ^2

$\triangleright$ no dependence upon energy (x)

$\triangleright$ the energy dependence enters only via the quantum evolution

Two gluons

- The intermediate gluon $(x_1,\, k_{1\perp})$ is not measured, but it acts as a source for the measured one $(x,\, k_\perp)$



$$x \ll 1,\ \Lambda^2 \ll k_\perp^2 \ll Q^2$$

$$x \ll x_1 \ll 1$$

$$\Lambda^2 \ll k_{1\perp}^2 \ll k_\perp^2$$

- N.B. The lifetime of the intermediate gluon is much larger:

$$\Delta t_1 \simeq \frac{2x_1 p_z}{k_{1\perp}^2} \gg \Delta t \simeq \frac{2x p_z}{k_\perp^2}$$

- The 2-gluon contribution to the gluon distribution measured at x and Q^2

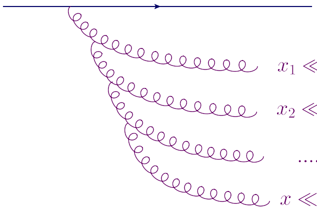
$$xG^{(1)}(x,Q^2) = \frac{\alpha_s C_F}{\pi}\int_{\Lambda^2}^{Q^2}\frac{\mathrm{d}k_\perp^2}{k_\perp^2}\,\frac{\alpha_s N_c}{\pi}\int_x^1\frac{\mathrm{d}x_1}{x_1}\int_{\Lambda^2}^{k_\perp^2}\frac{\mathrm{d}k_{1\perp}^2}{k_{1\perp}^2}$$

$$= xG^{(0)}(x,Q^2)\,\frac{1}{2}\,\frac{\alpha_s N_c}{\pi}\ln\frac{Q^2}{\Lambda^2}\ln\frac{1}{x}$$

The double logarithmic approximation

- When $\bar{\alpha} Y \rho \gtrsim 1 \implies$ need for all-order resummation

$$\bar{\alpha} \equiv \frac{\alpha_s N_c}{\pi}, \quad Y \equiv \ln\frac{1}{x}, \quad \rho \equiv \ln\frac{Q^2}{\Lambda^2}$$



- Strong ordering in both x (decreasing):

$$x \ll x_n \ll x_{n-1} \cdots \ll x_1 \ll 1$$

- ... and $k_\perp$ (increasing):

$$Q^2 \gg k_\perp^2 \gg k_{n\perp}^2 \cdots \gg k_{1\perp}^2 \gg \Lambda^2$$

$$\bar{\alpha}^n \int_x^1 \frac{\mathrm{d}x_n}{x_n} \int_{x_n}^1 \frac{\mathrm{d}x_{n-1}}{x_{n-1}} \cdots \int_{x_2}^1 \frac{\mathrm{d}x_n}{x_n} = \frac{1}{n!} \left(\bar{\alpha} Y\right)^n$$

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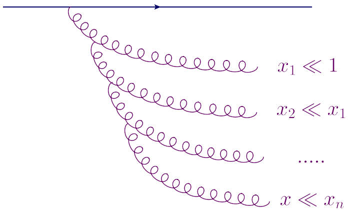

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$$Q^2 \gg k_\perp^2 \gg k_{n\perp}^2 \cdots \gg k_{1\perp}^2 \gg \Lambda^2$$

- After summing over cascades with any number $n \geq 0$ of intermediate gluons:

$$xG(x, Q^2) = xG^{(0)}(x, Q^2) \sum_{n \geq 0} \frac{(\bar{\alpha} Y \rho)^n}{(n!)^2} = xG^{(0)}(x, Q^2)\, \mathrm{I}_0(2\sqrt{\bar{\alpha} Y \rho})$$

- $\mathrm{I}_0(x)$: modified Bessel function of rank zero

The double logarithmic approximation

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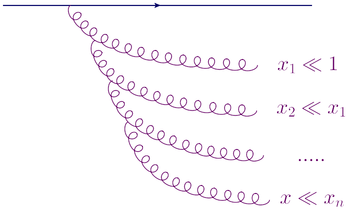

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- ... and $k_\perp$ (increasing):

$$Q^2 \gg k_\perp^2 \gg k_{n\perp}^2 \cdots \gg k_{1\perp}^2 \gg \Lambda^2$$

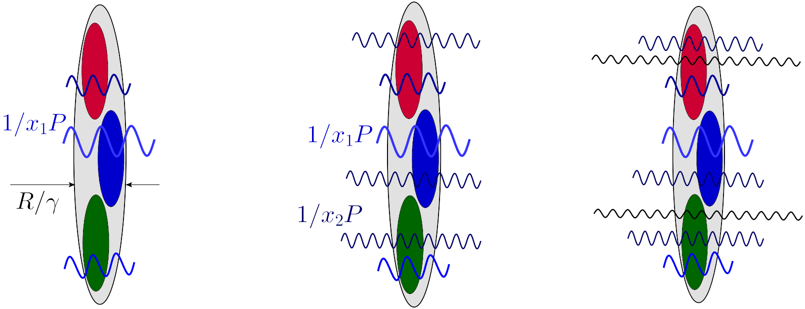
- Asymptotic behavior at small-x and large-Q^2 : $\bar{\alpha} Y \rho \gg 1$

$$xG(x, Q^2) \propto \mathrm{e}^{2\sqrt{\bar{\alpha} Y \rho}} \propto \exp\left\{2\sqrt{\bar{\alpha} \ln\frac{1}{x} \ln\frac{Q^2}{\Lambda^2}}\right\}$$

- Rapid increase with both $1/x$ and Q^2: faster than any power

Gluon evolution at small x

- A gluon with $\vec{\mathbf{k}} = (\boldsymbol{k}_\perp,\, k_z = xP)$ has a longitudinal extent $\Delta z \sim 1/xP$ and occupies a transverse area $\Delta x_\perp^2 \sim 1/k_\perp^2$
  - small-x gluons can easily overlap in the longitudinal direction
  - to actually overlap, their transverse momenta need to be small enough



- An energetic proton with $P = \gamma M$ and $\gamma \gg 1$ ("infinite momentum frame")
- DLA (generally, DGLAP) evolution maintains a dilute system of partons
  - rapid decrease in their transverse sizes $\Longrightarrow$ no possible overlap

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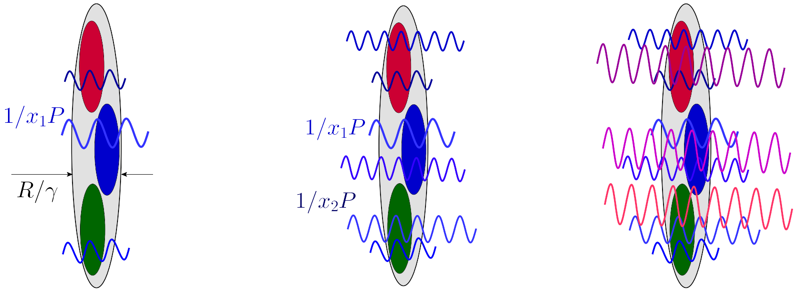


- BFKL evolution *(Balitsky, Fadin, Kuraev, Lipatov, 1974-78)*
  - decrease x at roughly fixed $k_\perp$: $\sum_n [\bar{\alpha}\ln(1/x)]^n \Longrightarrow$ increasing density
  - leading logarithmic approximation (LLA) at small x

Gluon occupancy

- Overlapping gluons can interact with each other
- What matter is not the density (# of gluons per unit transverse area)
- ... but the gluon occupation number (or 'packing factor')

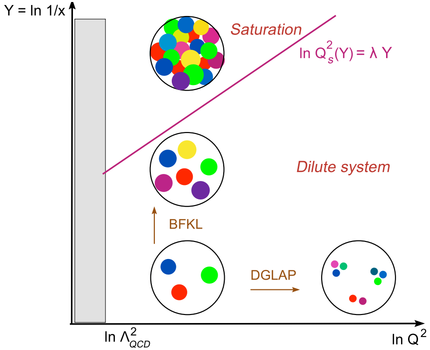
- takes into account their $\perp$ size

$$n(x, \boldsymbol{k}_\perp) \equiv x \frac{\mathrm{d}N_{\text{gluon}}}{\mathrm{d}x\,\mathrm{d}^2\boldsymbol{k}_\perp\,\mathrm{d}^2\boldsymbol{x}_\perp}$$

- a simple estimate

$$n(x, Q^2) \simeq \frac{1}{Q^2} \times \frac{xG(x, Q^2)}{\pi R^2}$$

- dilute systems have $n \ll 1$



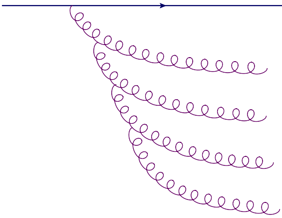
- When $n \gtrsim 1$, gluons overlap, but their interactions are still suppressed by α_s

Gluon saturation

- HERA data suggest:

$$n(x,Q^2) \simeq \frac{xG(x,Q^2)}{R^2\,Q^2} \sim \frac{1}{x^\lambda}$$

with $\lambda = 0.2 \div 0.3$.



- Consistent with BFKL evolution: $Y \equiv \ln(1/x)$

$$\frac{\partial n}{\partial Y} = \omega \alpha_s n \quad \Longrightarrow \quad n \propto \mathrm{e}^{\omega \alpha_s Y}$$

  - Linear equation: after being emitted, gluons do not interact with each other, but merely act as sources for new gluons, with smaller values of x
  - A reasonable assumption if the gluon occupancy is not that high: $n \lesssim 1$
  - When $n \sim 1/\bar{\alpha}$, gluons self interactions become of $\mathcal{O}(1)$: what happens?
  - A related question: can the gluon distribution $xG(x,Q^2)$ keep growing when x becomes arbitrarily small ?

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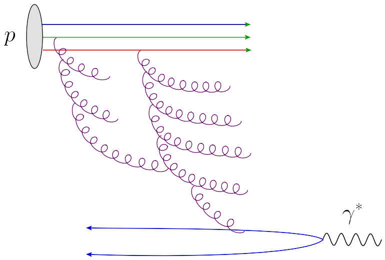


- Non-linear evolution (GLR)

$$\frac{\partial n}{\partial Y} = \alpha_s n - \alpha_s^2 n^2 \sim 0 \;\; \text{when } n \sim \frac{1}{\alpha_s}$$

- The idea of gluon saturation *(L. Gribov, Levin, Ryskin, 1982)*
  - gluon recombination $(gg \to g)$ becomes important and equilibrates gluon splitting $(g \to gg)$
  - gluon occupation number saturates at a value $n \sim 1/\bar{\alpha}$
- The actual physics is more complicated than just "gluon recombination"
- GLR equation: cartoon version of the non-linear evolution in QCD at small x

The saturation momentum

- The non-linear physics introduces a characteristic transverse momentum

  ▷ the transverse momentum below which gluons are at saturation

$$n(x, Q^2) \simeq \frac{xG(x, Q^2)}{R^2\, Q^2} \sim \frac{1}{\bar{\alpha}} \quad \text{when} \quad Q^2 \lesssim Q_s^2(x)$$

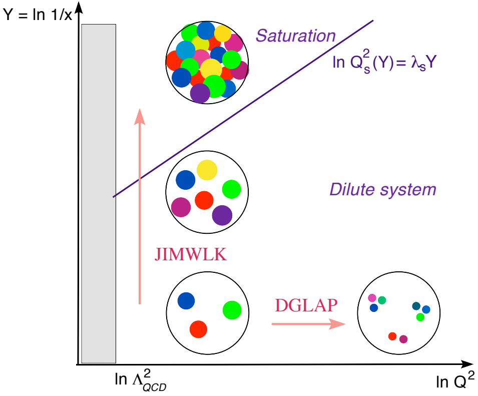
- The saturation momentum

$$Q_s^2(x) \simeq \bar{\alpha}\, \frac{xG(x, Q_s^2)}{R^2} \sim \frac{1}{x^{\lambda_s}}$$

- Gluon density in transverse plane

- $Q_s^2(x)$ rises with $1/x$, hence with the energy: $\lambda_s \simeq 0.2 \div 0.3$

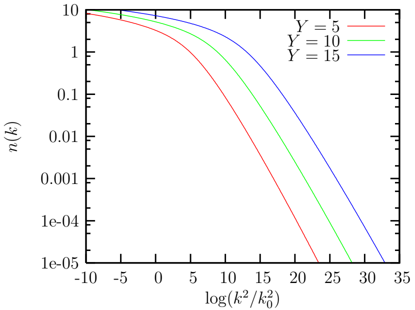
- Non-linear evolution: BK-JIMWLK



The saturation front

- A more precise definition of the gluon occupation number

$$n(Y, \boldsymbol{k}_\perp) \equiv \frac{(2\pi)^3}{2N_g} \frac{\mathrm{d}N_{\rm gluon}}{\mathrm{d}Y\, \mathrm{d}^2\boldsymbol{k}_\perp \mathrm{d}^2\boldsymbol{x}_\perp}\,, \qquad N_g \equiv N_c^2 - 1$$



- One roughly has

$$n(x, k_\perp) \simeq \begin{cases} \dfrac{1}{\bar\alpha} & \text{for } \ k_\perp \lesssim Q_s(x)\,, \\[2ex] \dfrac{1}{\bar\alpha}\,\dfrac{Q_s^2(x)}{k_\perp^2} & \text{for } \ k_\perp \gg Q_s(x)\,. \end{cases}$$

- Occupation numbers are rapidly decreasing when increasing $k_\perp$ above Q_s
- $Q_s(x)$ is also the typical transverse momentum for gluons with $x \ll 1$

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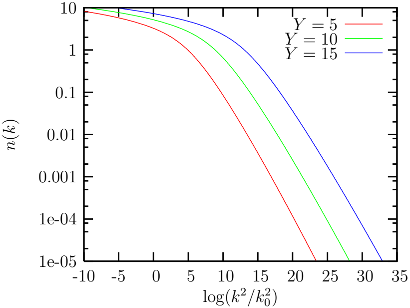

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$$Q_s^2(x) \propto \frac{1}{x^{\lambda_s}} \implies \ln Q_s^2(Y) \simeq \lambda_s Y$$

- A front which progresses towards larger $k_\perp$ when increasing $Y = \ln(1/x)$

- The saturation exponent λ_s : the speed of the front in logarithmic units

The saturation front

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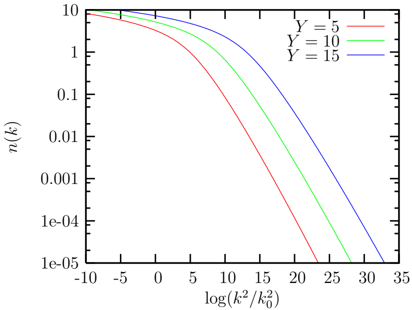

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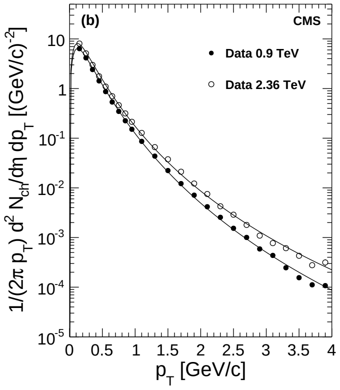
$$Q_s^2(x) \propto \frac{1}{x^{\lambda_s}} \implies \ln Q_s^2(Y) \simeq \lambda_s Y$$

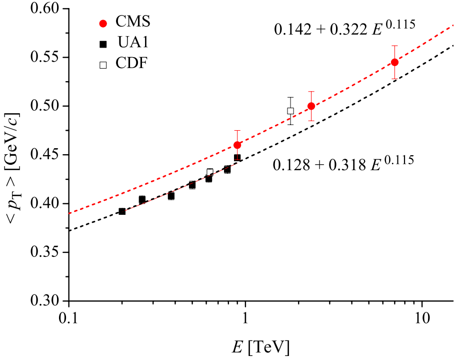
- To have large occupation numbers $n \sim \frac{1}{\bar\alpha} \gg 1$, one needs weak coupling

- For sufficiently large Y: $Q_s^2(Y) \gg \Lambda^2_{\rm QCD} \implies \alpha_s\big(Q_s(Y)\big) \ll 1 \implies$ pQCD

Average $p_\perp$ in pp (LHC) and $p\bar{p}$ (Tevatron)

- The saturated gluons are released by the collision and fragment into hadrons in the final state
- Typical transverse momentum: $\langle p_T \rangle \propto Q_s(x) \sim E^{\lambda_s/2}$ $(E \equiv \sqrt{s})$





(McLerran and Praszalowicz, 2010)

Multiplicity : energy dependence

- Particle multiplicity $\mathrm{d}N/\mathrm{d}\eta$: number of hadrons per unit rapidity near $\eta = 0$
- Controlled by soft hadrons ($p_\perp \lesssim 1$ GeV), hence by saturated gluons

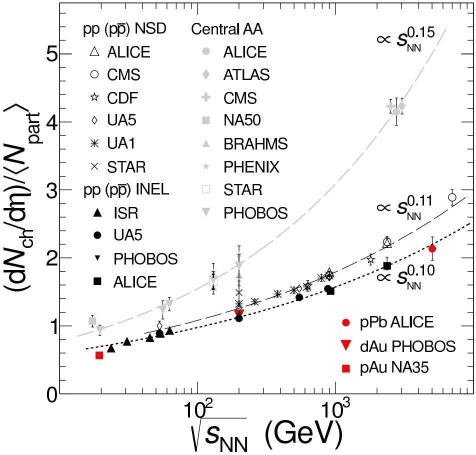
$$\frac{\mathrm{d}N}{\mathrm{d}\eta} \propto xG(x, Q_s^2) \propto Q_s^2(x)$$

▷ which value for x ?

$$x \simeq \frac{k_\perp}{\sqrt{s}} \quad \& \quad k_\perp \sim Q_s$$

$$Q_s^2(x) \propto \frac{1}{x^{\lambda_s}} \sim s^{\frac{\lambda_s}{2+\lambda_s}}$$

$$\lambda_s \simeq 0.2 \div 0.3$$



The nuclear oomph factor

- For a single proton (N_c valence quarks) and to leading order (no evolution)

$$n(Y,\boldsymbol{k}_\perp) \equiv \frac{(2\pi)^3}{2N_g}\,\frac{\alpha_s C_F N_c}{\pi^2 R^2 k_\perp^2} \quad \Longrightarrow \quad Q_{0p}^2 = \frac{2\alpha_s^2 N_c}{R^2} \sim (0.2\,\mathrm{GeV})^2$$

$\triangleright$ $Q_{0p} \sim \Lambda_{\rm QCD}$ (confinement) $\Longrightarrow$ dilute system, no gluon saturation

- Large nucleus ($A \gg 1$): Au: $A = 197$ (RHIC) or Pb: $A = 207$ (LHC)
- Incoherent superposition of A nucleons *(McLerran–Venugopalan model, '94)*

$$Q_{0A}^2 = \frac{2\alpha_s^2 A N_c}{R_A^2} = A^{1/3} Q_{0p}^2 \sim (0.5\,\mathrm{GeV})^2$$

- The saturation momentum of a large nucleus is semi-hard already at not so small values of x (say, for $x_0 \lesssim 0.05$)

$\frac{C_F N_c}{N_g} = 1/2$; $R_A^2 = A^{2/3} R^2$; $R \sim \frac{1}{\Lambda_{\rm QCD}} \sim 1$ fm; 1 fm $\times$ 1 GeV $\simeq 5$

Gluon evolution towards saturation (heuristically)

- The energy dependence of the gluon distribution, hence of the saturation momentum, arises exclusively from the quantum evolution
- In presence of saturation evolution equations are non-linear (BK-JIMWLK)
- The Y-dependence of $Q_s^2(Y)$ can be also computed from the linearized (BFKL) equation, supplemented with a saturation condition
- Just for simplicity, let's use the double-log approximation ($\rho \equiv \ln \frac{k_\perp^2}{Q_0^2}$)

$$n(Y, k_\perp^2) = n^{(0)}(k_\perp^2)\, \mathrm{I}_0(2\sqrt{\bar{\alpha} Y \rho}) \simeq \frac{1}{\bar{\alpha}}\, \mathrm{e}^{-\rho}\, \mathrm{e}^{2\sqrt{\bar{\alpha} Y \rho}}$$

$$n(Y, k_\perp^2 = Q_s^2(Y)) = \frac{1}{\bar{\alpha}} \implies \rho_s(Y) = 2\sqrt{\bar{\alpha} Y \rho_s(Y)} = 4\bar{\alpha} Y$$

- Develop for ρ close to (but above) $\rho_s(Y)$

$$n(Y, \rho) \simeq \frac{1}{\bar{\alpha}}\, \mathrm{e}^{-\frac{1}{2}(\rho - \rho_s(Y))}\, \mathrm{e}^{-\frac{(\rho - \rho_s(Y))^2}{16\bar{\alpha} Y}}$$

Gluon evolution towards saturation (heuristically)

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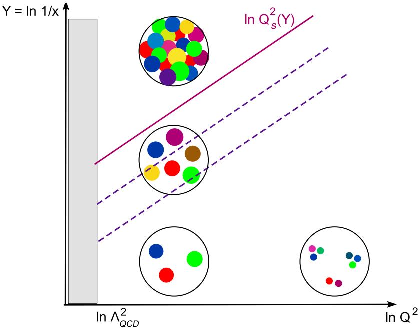
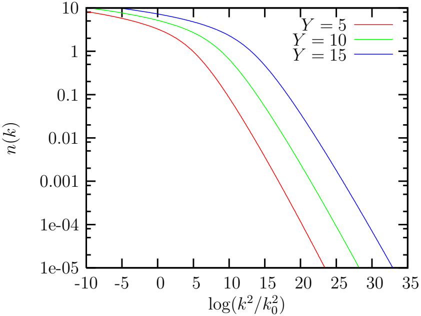
- "Geometric scaling" within a window $\rho - \rho_s(Y)$ which extends with Y

$$n(Y, \rho) \simeq \frac{1}{\bar\alpha}\, \mathrm{e}^{-\frac{1}{2}(\rho - \rho_s(Y))} = \frac{1}{\bar\alpha}\left[\frac{Q_s^2(Y)}{k_\perp^2}\right]^{\frac{1}{2}} \quad \text{if} \quad \rho - \rho_s \ll \sqrt{16\bar\alpha Y}$$

The traveling wave

- The occupation number 'scales' as a function of $Q_s^2(Y)/k_\perp^2$
  - instead of 2 independent variables ρ and Y, there is only one: the deviation $\rho - \rho_s(Y)$ from the saturation line

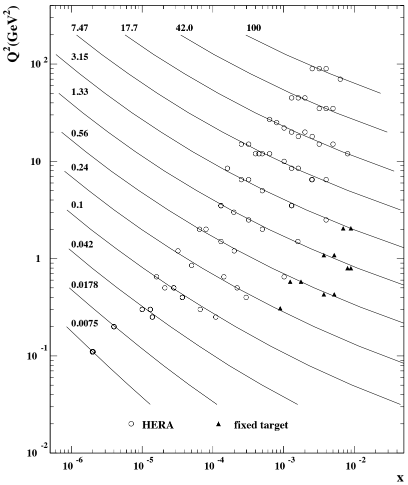


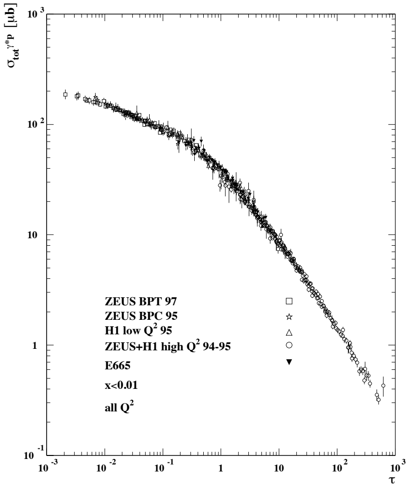


- $n(Y, \rho) \simeq n(\rho - \lambda_s Y)$: the front propagates at speed λ_s, without distorsion
- "Anomalous dimension" built by evolution: $Q_s^2/k_\perp^2 \rightarrow [Q_s^2/k_\perp^2]^\gamma$, $\gamma = 1/2$

- DIS cross–section at HERA *(Staśto, Golec-Biernat, Kwieciński, 2000)*

$$\sigma(x, Q^2) \ \text{vs.}\ \tau \equiv Q^2/Q_s^2(x) \propto Q^2/x^{0.3} : \ x \leq 0.01, \quad Q^2 \leq 450 \text{ GeV}^2$$



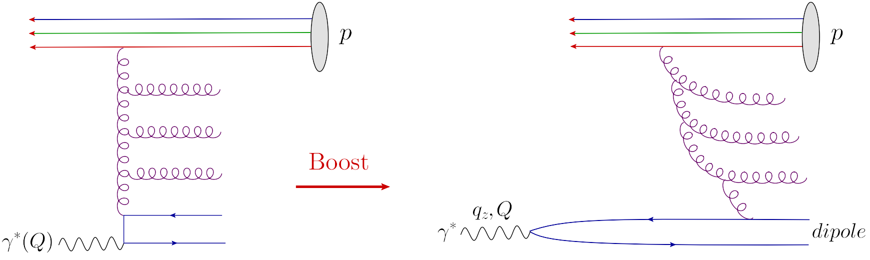


An intermediate check list

- So far, a lot of hand-waving arguments !
- We brought physical motivations and qualitative arguments in favor of:
  - the evolution of the gluon distribution towards higher density with increasing energy/decreasing x
  - the emergence of new collective phenomena like gluon saturation and multiple scattering
  - possible consequences of the phenomena such as geometric scaling
- To which extent is this physics grounded in pQCD ? What can we actually compute and how ?
- In what follows, we shall discuss high-energy factorization(s)
  - first for "dilute-dense": DIS and pA (proton-nucleus) collisions
  - eventually, also for "dense-dense": AA (nucleus-nucleus) collisions
- ... and the high energy evolution in the leading logarithmic approximation
  - Balitsky-Kovchegov, JIMWLK equations

DIS at high energy: the dipole frame

- DIS at small x: the evolution involves only gluons but γ^* couples to a sea quark (or antiquark) produced by the last gluon in the cascade
- Boost to a frame where γ^* is energetic ($q_0 \simeq q_z \gg Q$):
  - the $q\bar{q}$ can now be seen as a part of the photon wavefunction

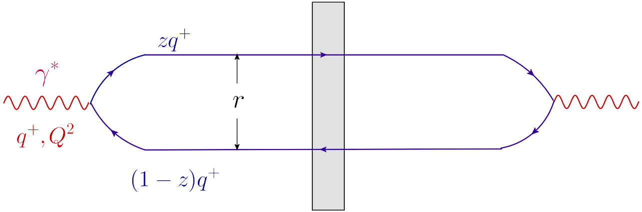


$$x \equiv \frac{Q^2}{2p \cdot q} \ll 1 \iff \Delta t \simeq \frac{2q_z}{Q^2} \gg \frac{1}{p_z}$$

- N.B. the physical picture & the factorization depend upon the Lorentz frame

Dipole factorization for DIS

- $\sigma_{\gamma^* p}$ = [probability for $\gamma^* \to q\bar{q}$] (QED) $\times$ $[\sigma_{q\bar{q}p}]$ (QCD)
- The $q\bar{q}$ pair is in a color singlet state: $\frac{1}{\sqrt{3}}\big(|R\bar{R}\rangle + |B\bar{B}\rangle + |G\bar{G}\rangle\big) = |\mathrm{dipole}\rangle$



$$\sigma_{\gamma^* p}(Q^2, x) = \int \mathrm{d}^2 r \int_0^1 \mathrm{d}z \, \left|\Psi_{\gamma^* \to q\bar{q}}(r, z; Q^2)\right|^2 \sigma_{\mathrm{dipole}}(r, x)$$

- γ^* wavefunction $\Psi_{\gamma^* \to q\bar{q}}(r, z; Q^2)$: computed in QED perturbation theory
- $r^2 \sim 1/Q^2$: dipole transverse size (dipole resolution in the transverse plane)
- The dipole cross-section σ_{dipole}: encodes the QCD scattering and evolution

Dipole factorization for DIS

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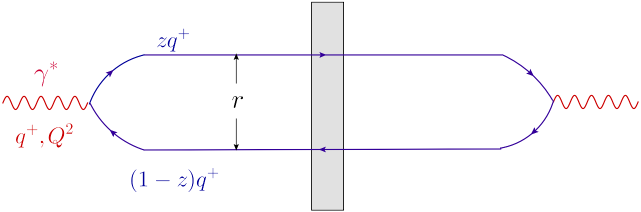
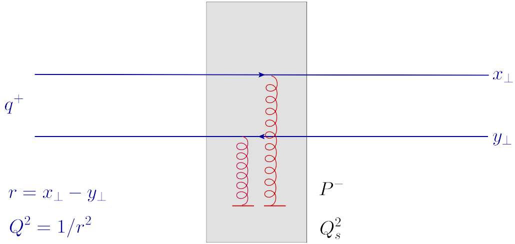


- Known for many years to leading order (LO) in α_s
  Nikolaev and Zakharov, 94; Al Mueller, 94;
  textbook by Forshaw and Ross ("QCD and the Pomeron")
- Recently extended to next-to-leading order (impact factor & evolution)
  (Balitsky and Chirilli, 2008-2013; Beuf, 2016-17)
- No general argument like OPE; no reason to be valid to all orders

Dipole-hadron scattering

- The dipole is a direct probe of the gluon distribution in the hadronic target
- A dipole couples to the electric (color) field: $V(\boldsymbol{r}) = g r^i t^a E_a^i$
- Forward scattering $\Longrightarrow$ two gluon exchange at LO ("single scattering")

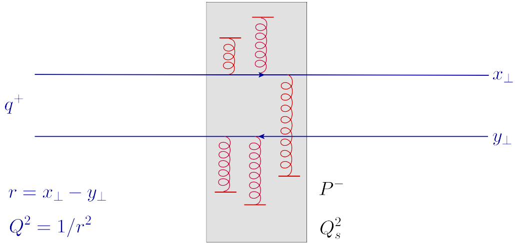
$$g^2 r^i r^j t^a t^b \langle E_a^i E_b^j \rangle = \frac{1}{2} g^2 C_F r^2 \langle E_a^i E_a^i \rangle \propto \alpha_s C_F r^2 \, xG(x, Q^2 = 1/r^2)$$



$$\sigma_{\rm dipole}(r, x) \simeq 2\pi^2 \alpha_s r^2 \frac{C_F}{N_g} \, xG(x, 1/r^2) \simeq 2\pi R^2 \left[r^2 Q_s^2(x) \right]$$

Multiple scattering

- A hadronic cross section cannot growth with E "much faster" than the geometric cross-section
  - $\sigma(E) \leq 2\pi R^2 \ln^2(E/M)$: Froissart bound
- Single scattering is a good approximation only so long as $rQ_s(x) \ll 1$



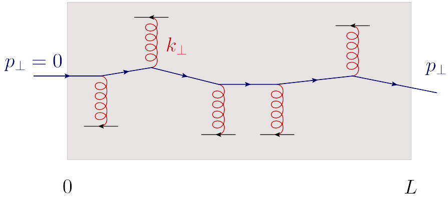
- When $r \gtrsim 1/Q_s(x)$, the scattering probes a dense gluon distribution
  - multiple scattering should become important

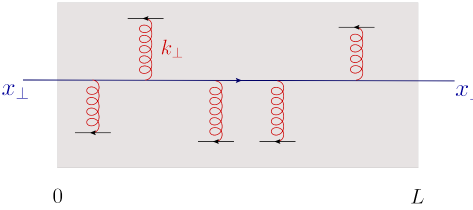
Eikonal approximation (1)

- Multiple scattering can be resummed in the eikonal approximation
  - the transverse coordinates $(\boldsymbol{x}_\perp, \boldsymbol{y}_\perp)$ of the quark/antiquark are not changed by the scattering off the shockwave

$$\Delta x_\perp \simeq v_\perp L \sim \frac{p_\perp}{q_z}\frac{1}{P_z} \ll \lambda_\perp \sim \frac{2}{p_\perp} \quad \text{or} \quad \frac{p_\perp^2}{2q_z P_z} \ll 1$$

  - automatic, since $p_\perp^2 < Q^2$ and $x = Q^2/(2q \cdot P) \ll 1$



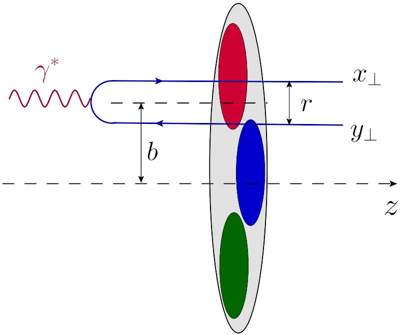


- Convenient to use the transverse coordinate representation *(cf. Markus)*

Eikonal approximation (2)

- Optical theorem: total cross-section = imaginary part of the forward scattering amplitude

$$\sigma_{\rm dipole}(r,x) = 2\int {\rm d}^2\boldsymbol{b}\; T(\boldsymbol{r},\,\boldsymbol{b},x)$$



- $\boldsymbol{r} = \boldsymbol{x}_\perp - \boldsymbol{y}_\perp$: dipole size
- $\boldsymbol{b} = (\boldsymbol{x}_\perp + \boldsymbol{y}_\perp)/2$: impact parameter
- $T(\boldsymbol{r},\,\boldsymbol{b},x) = 1 - \langle\hat{S}\rangle$: dipole amplitude
- $T \le 1$: unitarity bound

- S-matrix operator: $\hat{S} = {\rm T}\,{\rm e}^{{\rm i}\int {\rm d}^4x\,\mathcal{L}_{\rm int}(x)}$ with $\mathcal{L}_{\rm int}(x) = j_a^\mu(x)A_\mu^a(x)$
  - $j_a^\mu(x)$: color current of the dipole; $A_\mu^a(x)$: color field of the target
- One quark: $j_a^\mu(x) \simeq g v^\mu t^a \delta(z-t)\delta^{(2)}(\boldsymbol{x}_\perp - \boldsymbol{x}_\perp^0)$; $v^\mu = \frac{p^\mu}{p_0} = (1,0,0,1)$

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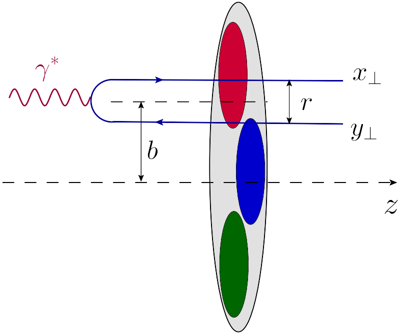

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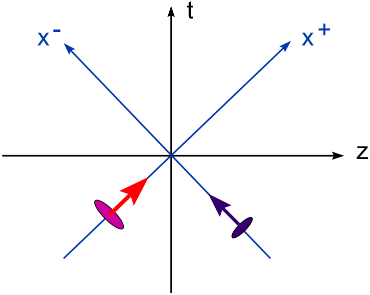
- A dipole: $j_a^\mu(x)\simeq gv^\mu t^a\delta(z-t)\Big\{\delta^{(2)}(\boldsymbol{x}_\perp-\boldsymbol{x}_\perp^0)-\delta^{(2)}(\boldsymbol{x}_\perp-\boldsymbol{y}_\perp^0)\Big\}$

Eikonal approximation (3)

- $v^\mu = (1,0,0,1) \Longrightarrow v^\mu A^a_\mu = A^0_a - A^z_a$

$$\hat{S}_q(\boldsymbol{x}_\perp) \;=\; \mathrm{T}\,\mathrm{e}^{\mathrm{i}\int \mathrm{d}t\, t^a (A^0_a - A^z_a)(t,\boldsymbol{x}_\perp, z=t)} \equiv V(\boldsymbol{x}_\perp)$$

- Convenient to use light–cone coordinates and momenta



$$x^\pm = \frac{1}{\sqrt{2}}(t \pm z)$$

$$p^\pm = \frac{1}{\sqrt{2}}(p^0 \pm p^z)$$

$$p\cdot x = p^+x^- + p^-x^+ - \boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp$$

$$\mathrm{d}t\mathrm{d}z = \mathrm{d}x^+\mathrm{d}x^-$$

- Ultrarelativistic right mover (the dipole) : $v^\mu = \sqrt{2}\delta^{\mu +}$, $v^\mu A^a_\mu = \sqrt{2}A^-_a$
- Left mover (the target): the roles of x^+ and x^- get interchanged

Eikonal approximation (3)

- $v^\mu = (1,0,0,1) \Longrightarrow v^\mu A_\mu^a = A_a^0 - A_a^z$

$$\hat{S}_q(\boldsymbol{x}_\perp) = \mathrm{T}\, \mathrm{e}^{\mathrm{i}\int \mathrm{d}x^+\, t^a A_a^-(x^+,\boldsymbol{x}_\perp,x^-=0)} \equiv V(\boldsymbol{x}_\perp)$$

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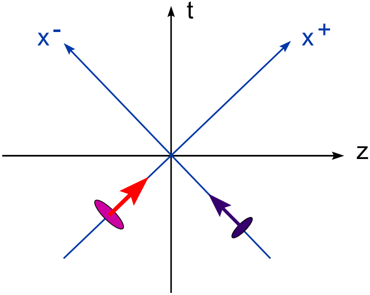

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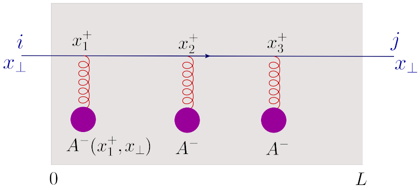
Wilson lines

$$V(\boldsymbol{x}_\perp) = \mathrm{T}\exp\left\{ig\int \mathrm{d}x^+ A_a^-(x^+,\boldsymbol{x}_\perp)t^a\right\}$$

- The fundamental degrees of freedom for scattering in QCD at high energies
- An exponential : multiple scattering is resummed to all orders
- $A_a^-(x)$: classical color field representing the small-x gluons in the target
- A color matrix (here, in the fundamental representation)
- A unitary matrix: $V(x_\perp)V^\dagger(x_\perp) = 1 \Longrightarrow$ a rotation of the quark color state



$$\Psi_i(x_\perp) \longrightarrow V_{ji}(x_\perp)\,\Psi_i(x_\perp)$$

- A time-ordered exponential: color matrices do not commute with each other

Wilson lines

$$V_N(\boldsymbol{x}_\perp) = \mathrm{e}^{\mathrm{i}g\epsilon A^-_N}\,\mathrm{e}^{\mathrm{i}g\epsilon A^-_{N-1}}\,\cdots\,\mathrm{e}^{\mathrm{i}g\epsilon A^-_1}\,\mathrm{e}^{\mathrm{i}g\epsilon A^-_0} \qquad \left(A^-_n \equiv A^-_a(x^+_n, \boldsymbol{x}_\perp)t^a\right)$$

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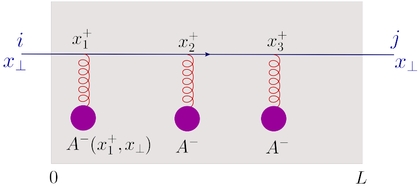

$$\Psi_i(x_\perp) \longrightarrow V_{ji}(x_\perp)\,\Psi_i(x_\perp)$$

- Best understood with a discretization of time: $x^+_n = n\epsilon$, $n = 0, 1, \cdots N$

The dipole S-matrix

- 2 Wilson lines: $V(\boldsymbol{x}_\perp)$ for the quark (q) and $V^\dagger(\boldsymbol{y}_\perp)$ for the antiquark $(\bar{q})$
  - an antiquark has charge $(-g)$ and propagates backwards in time
- Color singlet: the same color states for q and $\bar{q}$ before & after the scattering
  - sum over final color states & and average over the initial ones

$$\hat{S}_{\rm dipole}(\boldsymbol{x}_\perp, \boldsymbol{y}_\perp) = \frac{1}{N_c} V^\dagger_{ij}(\boldsymbol{y}_\perp) V_{ji}(\boldsymbol{x}_\perp) = \frac{1}{N_c} \mathrm{tr}\big(V(\boldsymbol{x}_\perp) V^\dagger(\boldsymbol{y}_\perp)\big)$$

- Perturbative expansion: multiple scattering series ... for a quark

$$\begin{aligned} V(\boldsymbol{x}_\perp) =& \, 1 + ig \int \mathrm{d}x^+ A^-_a(x^+, \boldsymbol{x}_\perp) t^a \\ & - \frac{g^2}{2} \int \mathrm{d}x^+ \int \mathrm{d}y^+ \big[\theta(x^+ - y^+) t^a t^b + \theta(y^+ - x^+) t^b t^a\big] A^-_a(x^+) A^-_a(y^+) \\ & + \mathcal{O}(g^3) \end{aligned}$$

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- Perturbative expansion: multiple scattering series ... and for the dipole

$$\hat{S}_{\rm dipole}(\boldsymbol{x}_\perp, \boldsymbol{y}_\perp) = 1 - \frac{g^2}{4N_c}\left[A^-_a(\boldsymbol{x}_\perp) - A^-_a(\boldsymbol{y}_\perp)\right]^2 + \mathcal{O}(g^3)$$

$$A^-_a(\boldsymbol{x}_\perp) \equiv \int {\rm d}x^+ A^-_a(x^+, \boldsymbol{x}_\perp), \quad {\rm tr}\, t^a = 0, \quad {\rm tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$$

- Color non-commutativity becomes important at $\mathcal{O}(g^3)$ and higher

The target averaging

- The color fields A_a^- are random fields whose correlators characterize the distribution of small-x gluons in the target (proton, nucleus)
- Observables are obtained after averaging over these fields
- E.g. A single scattering is probing a 2-point function:

$$A_a^-(\boldsymbol{x}_\perp) - A_a^-(\boldsymbol{y}_\perp) \simeq (x_\perp^i - y_\perp^i)\frac{\partial}{\partial b^i} A_a^-(\boldsymbol{b}_\perp) = r^i F_a^{i-}(\boldsymbol{b}_\perp):$$ chromo-electric field

- After taking the square and averaging over $A^- \Longrightarrow$ the gluon distribution

$$\int^{Q^2} \mathrm{d}^2\boldsymbol{k} \int_{\boldsymbol{x},\boldsymbol{y}} \mathrm{e}^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \big\langle F_a^{i-}(\boldsymbol{x}) \tilde{V}_{ab}(\boldsymbol{x},\boldsymbol{y}) F_b^{i-}(\boldsymbol{y})\big\rangle \propto xG(x,Q^2)$$

- To compute multiple scattering, one needs all the n-point functions

$$\langle S_{\boldsymbol{xy}}\rangle = \int [\mathcal{D}A^-]\, W[A^-]\, \frac{1}{N_c}\,\mathrm{tr}\big(V_{\boldsymbol{x}} V_{\boldsymbol{y}}^\dagger\big)[A^-]$$

  - $W[A^-]$: functional probability distribution (gauge-invariant)

The target averaging

- The color fields A^-_a are random fields whose correlators characterize the distribution of small-x gluons in the target (proton, nucleus)

- Observables are obtained after averaging over these fields

- E.g. A single scattering is probing a 2-point function:

$$A^-_a(\boldsymbol{x}_\perp) - A^-_a(\boldsymbol{y}_\perp) \simeq (x^i_\perp - y^i_\perp)\frac{\partial}{\partial b^i}A^-_a(\boldsymbol{b}_\perp) = r^i F^{i-}_a(\boldsymbol{b}_\perp) : \quad \text{chromo-electric field}$$

- After taking the square and averaging over A^- $\Longrightarrow$ the gluon distribution

$$\int^{Q^2} \mathrm{d}^2\boldsymbol{k} \int_{\boldsymbol{x},\boldsymbol{y}} \mathrm{e}^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \big\langle F^{i-}_a(\boldsymbol{x}) \tilde{V}_{ab}(\boldsymbol{x},\boldsymbol{y}) F^{i-}_b(\boldsymbol{y})\big\rangle \propto xG(x,Q^2)$$

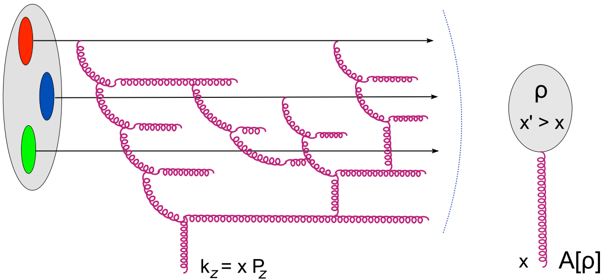
- To compute multiple scattering, one needs all the n-point functions

$$\langle S_{\boldsymbol{xy}}\rangle_Y = \int [\mathcal{D}A]\; W_Y[A]\; \frac{1}{N_c}\,\mathrm{tr}\big(V_{\boldsymbol{x}} V^\dagger_{\boldsymbol{y}}\big)[A]$$

  - includes the target evolution with $Y = \ln(1/x)$ (here $x = k^-/P^-$)

Color Glass Condensate

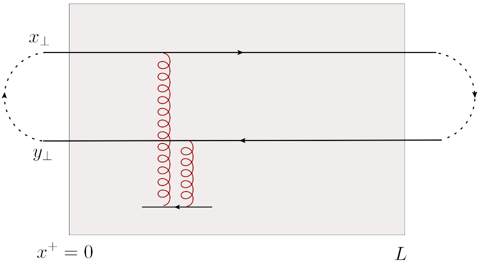
- Small–x gluons : classical color fields $A_a^\mu[\rho]$ radiated by a frozen distribution ρ_a of color charges representing partons with $x' \gg x$
  - lifetimes are strongly ordered by Lorentz time dilation: $\Delta t = 2xP_z/k_\perp^2$



- $W_Y[\rho]$: CGC weight function, built via renormalization group ("JIMWLK")
  - successively integrating out gluons in layers of x': high-energy evolution
  - initial condition at $x' \sim 0.1 \div 0.01$: McLerran-Venugopalan model

Dipole scattering in the MV model (1)

- A large nucleus $(A \gg 1)$ & not too small values of x $(\bar{\alpha} \ln 1/x \ll 1)$
- No quantum evolution: the only color sources are the $A \times N_c$ valence quarks
- Independent scatterings $\Longrightarrow$ the multiple scattering series exponentiates



$$S(r) = \mathrm{e}^{-T_0(r)}$$

$$T_0(r,x) \simeq \pi \alpha_s r^2 \frac{C_F}{N_g} \frac{x G_A^{(0)}(x, 1/r^2)}{R_A^2}$$

$$x G_A^{(0)}(x, Q^2) = \frac{\alpha_s C_F A N_c}{\pi} \ln \frac{Q^2}{\Lambda^2}$$

- The dipole scatters off all the quarks within an area $\sim r^2$ around its impact parameter $b_\perp$ $\Longrightarrow$ a tube with length $L = R_A = R A^{1/3}$
- No explicit $b_\perp$ dependence: target assumed to be homogeneous in $\perp$ plane

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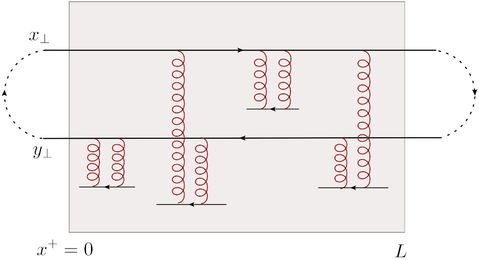


$$S(r) = \mathrm{e}^{-T_0(r)}$$

$$S(r) = \exp\left\{-\frac{r^2 Q_{0A}^2}{4} \ln \frac{1}{r^2\Lambda^2}\right\}$$

$$Q_{0A}^2 \equiv \frac{2\alpha_s^2 C_F A^{1/3}}{R^2}$$

- Q_{0A}^2: color charge squared of the valence quarks per unit area
- $\ln(1/r^2\Lambda^2)$: gluon exchanges within the range $r < \Delta x_\perp < 1/\Lambda$

Dipole scattering in the MV model (2)

$$S(r) = 1 - T(r) = \mathrm{e}^{-T_0(r)} = \exp\left\{-\frac{r^2 Q_{0A}^2}{4}\ln\frac{1}{r^2\Lambda^2}\right\}$$

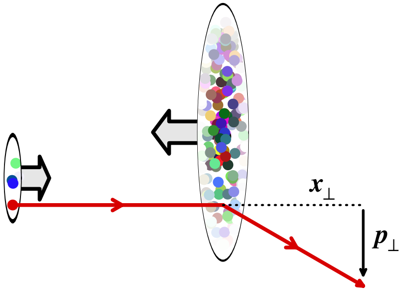
- $S(r) \to 1$ (i.e. $T(r) \to 0$) when $r \to 0$: color transparency
- $T(r) \leq 1$ for any r (and similarly $S(r) \leq 1$) : unitarity bound satisfied
- Scattering is weak, i.e. $T(r) \simeq T_0(r) \ll 1$, so long as the exponent is small.
- Scattering becomes strong, i.e. $T(r) \sim \mathcal{O}(1)$, when $T_0(r) \sim \mathcal{O}(1)$
  - the dipole should be large enough
- Saturation momentum Q_s: conventionally defined as $T_0(r) = 1$ for $\frac{2}{r} = Q_s$

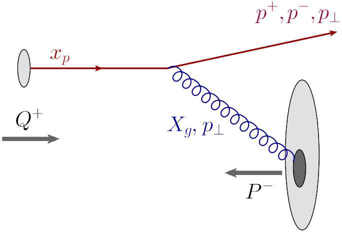
$$Q_s^2(A) = Q_{0A}^2 \ln\frac{Q_s^2(A)}{4\Lambda^2} \propto A^{1/3}\ln A^{1/3}$$

- Gluon saturation in the nucleus manifests as multiple scattering for the probe
  - typical scale for the onset of non-linear physics

Particle production in pA collisions

- A quark initially collinear with the proton acquires a transverse momentum $p_\perp \sim Q_s$ via multiple scattering off the saturated gluons
- Formally, a $2 \to 1$ process: $qg \to q$ (in contrast to collinear factoriz.: $2 \to 2$)





$$p^\pm = \frac{p^0 + p^z}{\sqrt{2}} = \frac{p_\perp}{\sqrt{2}}\,\mathrm{e}^{\pm\eta}$$

- $\eta = -\ln\tan\frac{\theta}{2}$: quark rapidity in the COM frame ($Q^+ = P^- = \sqrt{s/2}$)
- x_p : longitudinal fraction of the quark in the proton
- X_g : longitudinal fraction of the gluon in the target

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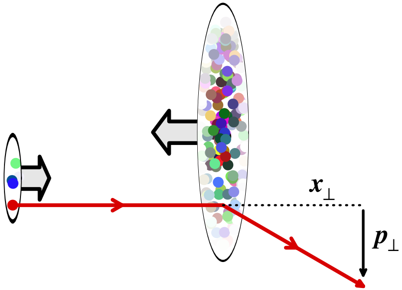


$$x_p \equiv \frac{p^+}{Q^+} = \frac{p_\perp}{\sqrt{s}}\,\mathrm{e}^{\eta}$$

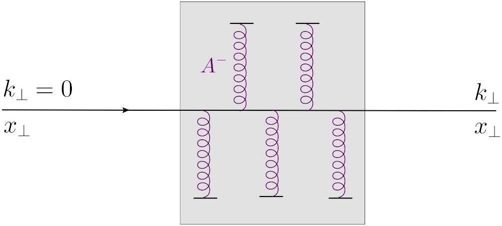
$$X_g \equiv \frac{p^-}{P^-} = \frac{p_\perp}{\sqrt{s}}\,\mathrm{e}^{-\eta}$$

$X_g \ll x_p$ when $\eta > 0$

- $\eta = -\ln\tan\frac{\theta}{2}$: quark rapidity in the COM frame ($Q^+ = P^- = \sqrt{s/2}$)
- x_p : longitudinal fraction of the quark in the proton
- X_g : longitudinal fraction of the gluon in the target

Multiple scattering

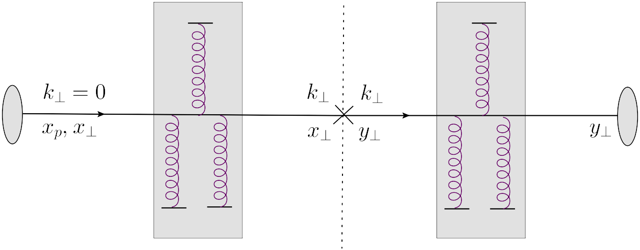
- The quark exchanges transverse momentum with the gluons in the target
  - a random process leading to a distribution in the final momentum $\boldsymbol{k}_\perp$
- Transverse momentum broadening can be studied in the eikonal approx.
  - fixed transverse coordinate, but the transverse momentum can vary



Amplitude: $$\mathcal{M}_{ij}(\boldsymbol{k}_\perp) \equiv \int \mathrm{d}^2\boldsymbol{x}_\perp \, \mathrm{e}^{-i\boldsymbol{x}_\perp\cdot\boldsymbol{k}_\perp} V_{ij}(\boldsymbol{x}_\perp)$$

Wilson line: $$V(\boldsymbol{x}_\perp) = \mathrm{T}\exp\left\{ig\int \mathrm{d}x^+ A_a^-(x^+,\boldsymbol{x}_\perp)t^a\right\}$$

Multiple scattering



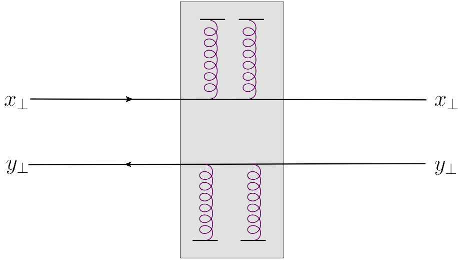
Amplitude: $$\mathcal{M}_{ij}(\boldsymbol{k}_\perp) \equiv \int \mathrm{d}^2\boldsymbol{x}_\perp \, \mathrm{e}^{-i\boldsymbol{x}_\perp\cdot\boldsymbol{k}_\perp} \, V_{ij}(\boldsymbol{x}_\perp)$$

Cross-section: $$\frac{\mathrm{d}\sigma}{\mathrm{d}\eta\mathrm{d}^2\boldsymbol{k}_\perp} \simeq x_p q(x_p, Q^2)\, \frac{1}{N_c} \left\langle \sum_{ij} |\mathcal{M}_{ij}(\boldsymbol{k}_\perp)|^2 \right\rangle_{X_g}$$

- Average over the color fields A^- in the target (CGC)
- Two Wilson lines at different transverse coordinates, traced over color

Dipole picture for pA collisions

- Equivalently: elastic S-matrix for a $q\bar{q}$ color dipole (here, a fictitious dipole)



$$S(\boldsymbol{x}_\perp, \boldsymbol{y}_\perp; X_g) \equiv \frac{1}{N_c} \left\langle \mathrm{tr}\left[V(\boldsymbol{x}_\perp) V^\dagger(\boldsymbol{y}_\perp)\right] \right\rangle_{X_g}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\eta \mathrm{d}^2\boldsymbol{k}} \simeq x_p q(x_p) \int \frac{\mathrm{d}^2\boldsymbol{x}\mathrm{d}^2\boldsymbol{y}}{(2\pi)^2} \, \mathrm{e}^{-\mathrm{i}(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{k}} \, S(\boldsymbol{x}, \boldsymbol{y}; X_g)$$

- Fourier transform $\mathcal{S}(\boldsymbol{k}, X_g)$ of the dipole S-matrix
  - "unintegrated gluon distribution", or "dipole TMD"

Momentum broadening in the MV model (1)

- Consider an incoming quark for simplicity and assume $S(\boldsymbol{x}, \boldsymbol{y}) = S(r)$

$$\frac{\mathrm{d}N}{\mathrm{d}^2\boldsymbol{k}} = \int \frac{\mathrm{d}^2\boldsymbol{r}}{(2\pi)^2}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\; \mathrm{e}^{-\frac{1}{4} r^2 Q_{0A}^2 \ln \frac{1}{r^2\Lambda^2}}$$

- Would-be a Gaussian integration ... if there were not for the logarithm
- Two interesting situations which allow for simple results
- "Typical values for $k_\perp$": $k_\perp \sim Q_s$, as transferred by multiple scattering
  - integral cut off at $r \sim 1/Q_s$ by the S-matrix $S(r)$
  - replace $1/r^2 \to Q_s^2$ within the argument of the log

$$\frac{\mathrm{d}N}{\mathrm{d}^2\boldsymbol{k}} \simeq \frac{1}{\pi Q_s^2(A)}\, \mathrm{e}^{-k_\perp^2/Q_s^2(A)}$$

  - a Gaussian distribution: random walk in $\boldsymbol{k}$

$$\langle k_\perp^2 \rangle \equiv \int \mathrm{d}^2\boldsymbol{k}\, k_\perp^2\, \frac{\mathrm{d}N}{\mathrm{d}^2\boldsymbol{k}} = Q_s^2(A) \propto L = RA^{1/3}$$

Momentum broadening in the MV model (1)

- Consider an incoming quark for simplicity and assume $S(\boldsymbol{x},\boldsymbol{y}) = S(r)$

$$\frac{\mathrm{d}N}{\mathrm{d}^2\boldsymbol{k}} = \int \frac{\mathrm{d}^2\boldsymbol{r}}{(2\pi)^2}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\; \mathrm{e}^{-\frac{1}{4}r^2 Q_{0A}^2 \ln\frac{1}{r^2\Lambda^2}}$$

- Would-be a Gaussian integration ... if there were not for the logarithm
- Two interesting situations which allow for simple results
- Large $k_\perp \gg Q_s$, as given by a single hard scattering
  - integral cut off at $r \sim 1/k_\perp$ by the exponential
  - $rQ_s \ll 1 \implies$ one can expand $S \simeq 1 - T_0$ (one scattering)

$$\frac{\mathrm{d}N}{\mathrm{d}^2\boldsymbol{k}} \simeq \frac{Q_{0A}^2}{\pi k_\perp^4}$$

  - an approximate version of the collinear factorization: $qq \to qq$

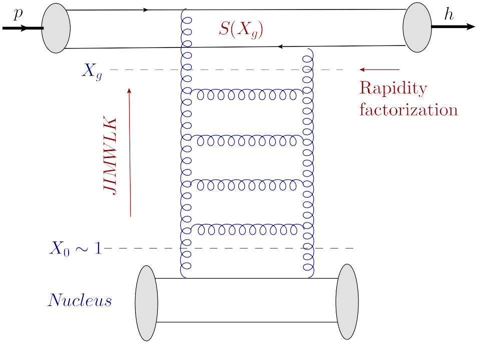
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\eta\mathrm{d}^2\boldsymbol{k}} \simeq x_p q(x_p)\,\frac{\alpha_s^2 C_F^2}{2k_\perp^4}\, x_A q_A(x_A)\,, \qquad x_A q_A(x_A) \equiv A N_c$$

High-energy factorization for pA ("hybrid")

- After scattering, the quark must "fragment into hadrons" : $D_{h/q}(z,\mu^2)$

$$\frac{\mathrm{d}\sigma_h}{\mathrm{d}\eta \mathrm{d}^2\boldsymbol{p}} = \int \frac{\mathrm{d}z}{z^2}\, x_p q(x_p,\mu^2) \left[\int_{\boldsymbol{x},\boldsymbol{y}} \mathrm{e}^{-\mathrm{i}(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{k}}\, S(\boldsymbol{x},\boldsymbol{y};X_g)\right] D_{h/q}(z,\mu^2)$$

- There is also a gluon-initiated channel, albeit less important when $x_p \sim \mathcal{O}(1)$



- Each soft gluon emission brings in a factor $\bar{\alpha}\ln(1/X_g)$

Hybrid factorization at leading order

(Kovchegov and Tuchin, 2002; Dumitru, Hayashigaki, and Jalilian-Marian, 2005)

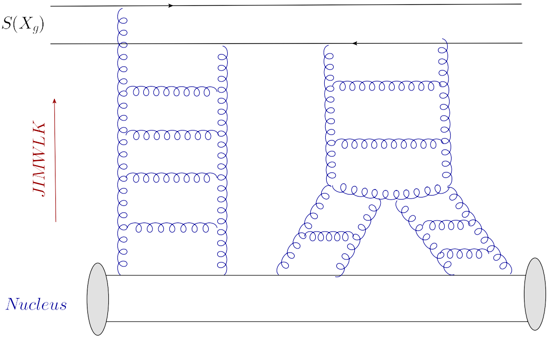
$$\frac{\mathrm{d}\sigma_h}{\mathrm{d}\eta\mathrm{d}^2\boldsymbol{p}} = \int \frac{\mathrm{d}z}{z^2}\, x_p q(x_p, \mu^2) \left[\int_{\boldsymbol{x},\boldsymbol{y}} \mathrm{e}^{-\mathrm{i}(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{k}}\, S(\boldsymbol{x},\boldsymbol{y}; X_g)\right] D_{h/q}(z, \mu^2)$$

- Collinear factorization for the incoming proton/outgoing hadron
  - LO DGLAP evolution for quark distribution/ fragmentation
- High-energy (CGC) factorization for the quark-nucleus scattering
  - LO JIMWLK (BK) for target gluon distribution (dipole S-matrix)
- Natural, but non-trivial already at leading order
  - one needs to prove the factorization of the two types of evolution
- The dipole picture is preserved by the high-energy evolution up to NLO *(Mueller and Munier, 2012; Chirilli, Xiao, and Yuan, 2012)*

The high-energy evolution of the target

$$\left\langle \mathrm{tr}\big(V_{\boldsymbol{x}} V_{\boldsymbol{y}}^{\dagger}\big)\right\rangle_{X_g} = \int [DA^-]\, W_{X_g}[A^-]\, \mathrm{tr}\big(V_{\boldsymbol{x}} V_{\boldsymbol{y}}^{\dagger}\big)$$

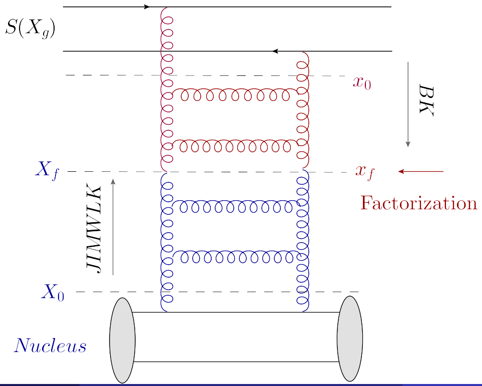
- JIMWLK: functional equation for the CGC weight function $W_{X_g}[A^-]$



- gluon emissions with smaller and smaller $X = p^-/P^-$, down to X_g
- non-linear effects in both the evolution (gluon saturation) and the collision (multiple scattering)

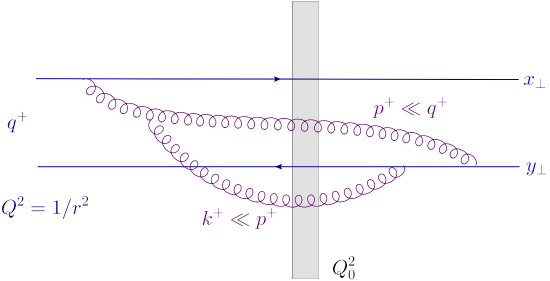
Shifting to projectile evolution

- The evolution can be shared between dilute projectile and dense target
  - JIMWLK evolution in $X = p^-/P^-$ from $X_0 \sim \mathcal{O}(1)$ down to X_f
  - BK evolution in $x = p^+/Q^+$ from $x_0 \sim x_p$ down to x_f
  - the evolution of the dipole is conceptually simpler
  - chose $x_f \sim X_g$ (i.e. $X_f \sim X_0$): target described by the MV model



Dipole evolution

- Probability $\sim \alpha_s \ln(1/x)$ for emitting a soft $(x \ll 1)$ gluon
  - $x \equiv k^+/q^+$: longitudinal momentum fraction for a right mover
- Gluons must be emitted and reabsorbed within the dipole (color neutrality)

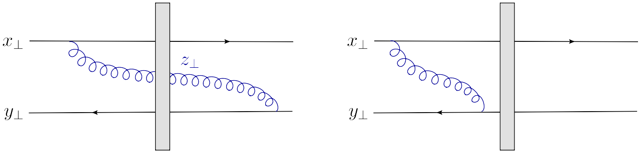


$$x_2 = \frac{k^+}{q^+} \ll x_1 = \frac{p^+}{q^+} \ll 1\,, \qquad Y = \ln\frac{1}{x_{\min}} = \ln\frac{s}{Q^2}$$

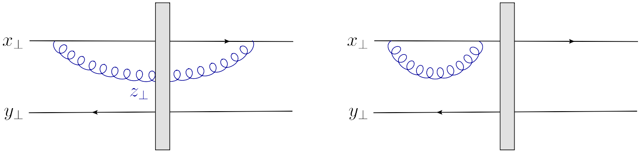
- Leading logarithmic approx: resum $(\bar{\alpha}Y)^n$ with $n \geq 1$

One step in the BK evolution (1)

- To construct the evolution equation, it is enough to look at the first emission
- The gluon can be exchanged between the quark and the antiquark



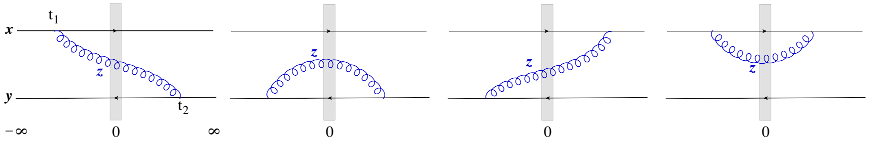
- ... or be emitted and reabsorbed by a same fermion ("self-energy graph")



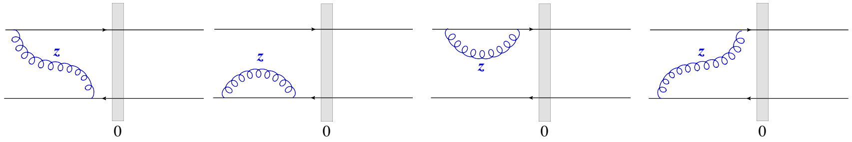
- In both cases, the gluon may cross the shockwave ("real contributions")
- ... or not ! "Virtual corrections", or "evolution in the initial/final state"

One step in the BK evolution (2)

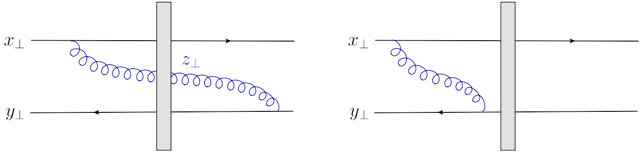
- And of course there are several possible permutations of the gluon vertices
- 'Real contributions': the soft gluon can interact with the shockwave
  - the system which scatters: a 3-parton system $(q\bar{q}g)$



- 'Virtual contributions': only the original $q\bar{q}$ dipole interacts



- Small step in rapidity: $\alpha_s \mathrm{d}Y \ll 1$



$$\mathrm{d}_1 S_Y(\boldsymbol{x},\boldsymbol{y}) = -\frac{\alpha_s}{\pi^2}\,\mathrm{d}Y \int \mathrm{d}^2\boldsymbol{z}\, \frac{(\boldsymbol{x}-\boldsymbol{z})^i}{(\boldsymbol{x}-\boldsymbol{z})^2}\,\frac{(\boldsymbol{y}-\boldsymbol{z})^i}{(\boldsymbol{y}-\boldsymbol{z})^2}$$

$$\left\langle \tilde{V}_{ab}(\boldsymbol{z})\,\frac{1}{N_c}\,\mathrm{tr}\Big(V(\boldsymbol{x})t^b\,V^\dagger(\boldsymbol{y})t^a\Big) - \frac{C_F}{N_c}\,\mathrm{tr}\big(V_{\boldsymbol{x}}V^\dagger_{\boldsymbol{y}}\big)\right\rangle_Y$$

- 'Real' term: the gluon emitted at $\boldsymbol{x}$ hits the shockwave a $\boldsymbol{z}$

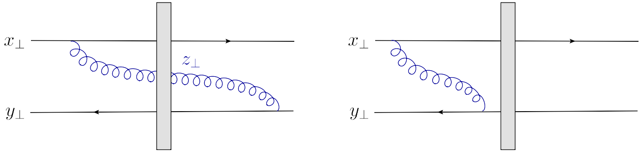
$$g\frac{(\boldsymbol{x}-\boldsymbol{z})^i}{(\boldsymbol{x}-\boldsymbol{z})^2} \;:\quad \text{amplitude for gluon emission and propagation from } \boldsymbol{x} \text{ to } \boldsymbol{z}$$

- Wilson line for the gluon at $\boldsymbol{z}$ in adjoint representation: $(T^a)_{bc} = if^{abc}$

$$\tilde{V}(\boldsymbol{z}) = \mathrm{T}\exp\left\{ig\int \mathrm{d}x^+ A^-_a(x^+,\boldsymbol{x}_\perp)T^a\right\}$$

One step in the BK evolution (3)

- Small step in rapidity: $\alpha_s \mathrm{d}Y \ll 1$



$$\mathrm{d}_1 S_Y(\boldsymbol{x},\boldsymbol{y}) = -\frac{\alpha_s}{\pi^2}\,\mathrm{d}Y \int \mathrm{d}^2\boldsymbol{z}\, \frac{(\boldsymbol{x}-\boldsymbol{z})^i}{(\boldsymbol{x}-\boldsymbol{z})^2}\,\frac{(\boldsymbol{y}-\boldsymbol{z})^i}{(\boldsymbol{y}-\boldsymbol{z})^2}$$

$$\left\langle \tilde{V}_{ab}(\boldsymbol{z})\,\frac{1}{N_c}\,\mathrm{tr}\Big(V(\boldsymbol{x})t^b\,V^\dagger(\boldsymbol{y})t^a\Big) - \frac{C_F}{N_c}\,\mathrm{tr}\big(V_{\boldsymbol{x}}V^\dagger_{\boldsymbol{y}}\big)\right\rangle_Y$$

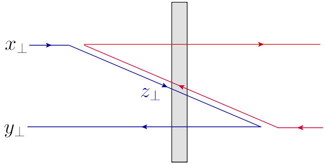
- 'Real' term: the gluon emitted at $\boldsymbol{x}$ hits the shockwave a $\boldsymbol{z}$

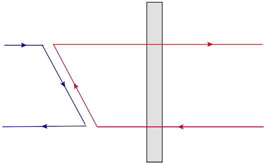
  $g\dfrac{(\boldsymbol{x}-\boldsymbol{z})^i}{(\boldsymbol{x}-\boldsymbol{z})^2}$: amplitude for gluon emission and propagation from $\boldsymbol{x}$ to $\boldsymbol{z}$

- Not a closed equation: evolution couples $S_{q\bar{q}}(\boldsymbol{x},\boldsymbol{y})$ to $S_{q\bar{q}g}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$
  - not a surprise: one additional gluon that is measured by the scattering

BK evolution at large N_c

- A closed evolution equation can be obtained in the multi-color limit $N_c \to \infty$
  - finite-N_c corrections are suppressed as $1/N_c^2 \lesssim 10\%$ for $N_c = 3$
- At large N_c, a gluon can be replaced by a quark-antiquark pair
  - gluon emission by a dipole $\approx$ dipole splitting into 2 dipoles





$$\mathrm{d}_1 S_Y(\boldsymbol{x},\boldsymbol{y}) \simeq -\frac{\alpha_s N_c}{2\pi^2}\mathrm{d}Y \int_{\boldsymbol{z}} \frac{(\boldsymbol{x}-\boldsymbol{z})^i}{(\boldsymbol{x}-\boldsymbol{z})^2}\,\frac{(\boldsymbol{y}-\boldsymbol{z})^i}{(\boldsymbol{y}-\boldsymbol{z})^2}\Big\{S_Y(\boldsymbol{x},\boldsymbol{z})\,S_Y(\boldsymbol{z},\boldsymbol{y}) - S_Y(\boldsymbol{x},\boldsymbol{y})\Big\}$$

- At large N_c, expectation values of colorless operators factorize

$$\left\langle \frac{\mathrm{tr}\big(V_{\boldsymbol{x}}V_{\boldsymbol{z}}^\dagger\big)}{N_c}\,\frac{\mathrm{tr}\big(V_{\boldsymbol{z}}V_{\boldsymbol{y}}^\dagger\big)}{N_c}\right\rangle_Y \simeq S_Y(\boldsymbol{x},\boldsymbol{z})\,S_Y(\boldsymbol{z},\boldsymbol{y})$$

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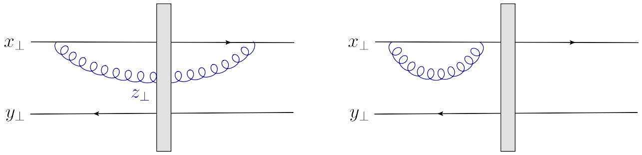
  - finite-N_c corrections are suppressed as $1/N_c^2 \lesssim 10\%$ for $N_c = 3$
- At large N_c, a gluon can be replaced by a quark-antiquark pair
  - gluon emission by a dipole $\approx$ dipole splitting into 2 dipoles
- Similar manipulations for self-energy graphs (opposite sign, due to $\bar{q} \to q$)



$$\mathrm{d}_2 S_Y(\boldsymbol{x},\boldsymbol{y}) \simeq \frac{\alpha_s N_c}{2\pi^2}\,\mathrm{d}Y \int_{\boldsymbol{z}} \frac{1}{(\boldsymbol{x}-\boldsymbol{z})^2}\Big\{S_Y(\boldsymbol{x},\boldsymbol{z})\,S_Y(\boldsymbol{z},\boldsymbol{y}) - S_Y(\boldsymbol{x},\boldsymbol{y})\Big\}$$

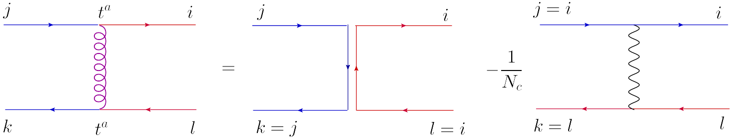
Fierz identities & the large N_c limit

- Fierz identities $\Longrightarrow$ all color structures in terms of fundamental Wilson lines

$$\tilde{V}^{ab}(\boldsymbol{z}) \;=\; 2\,\mathrm{tr}\left(t^a\,V(\boldsymbol{z})\,t^b\,V^\dagger(\boldsymbol{z})\right)$$

- SU(N_c) representations: $\mathbf{8} = \mathbf{3}\times\overline{\mathbf{3}} - \mathbf{1}$, $\quad \mathbf{N}_c^2 - 1 = \mathbf{N}_c \times \overline{\mathbf{N}}_c - \mathbf{1}$
  - a gluon = a quark-antiquark pair minus a "photon"

$$t^a_{ij}\,t^a_{kl} \;=\; \frac{1}{2}\delta_{il}\,\delta_{jk} \;-\; \frac{1}{2N_c}\delta_{ij}\,\delta_{kl}$$



$$\tilde{V}_{ab}(\boldsymbol{z})\,\frac{1}{N_c}\mathrm{tr}\big(V(\boldsymbol{x})t^b\,V^\dagger(\boldsymbol{y})t^a\big) = \frac{N_c}{2}\left\{\frac{\mathrm{tr}\big(V_{\boldsymbol{x}}V^\dagger_{\boldsymbol{z}}\big)}{N_c}\,\frac{\mathrm{tr}\big(V_{\boldsymbol{z}}V^\dagger_{\boldsymbol{y}}\big)}{N_c} - \frac{1}{N_c^2}\,\frac{\mathrm{tr}\big(V_{\boldsymbol{x}}V^\dagger_{\boldsymbol{y}}\big)}{N_c}\right\}$$

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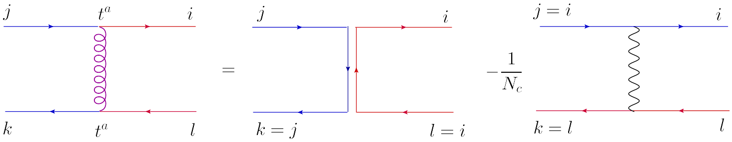

$$\left\langle \tilde{V}_{ab}(\boldsymbol{z})\,\frac{1}{N_c}\mathrm{tr}\big(V(\boldsymbol{x})t^b\,V^\dagger(\boldsymbol{y})t^a\big)\right\rangle_Y = \frac{N_c}{2}\left\langle \frac{\mathrm{tr}\big(V_{\boldsymbol{x}}V^\dagger_{\boldsymbol{z}}\big)}{N_c}\,\frac{\mathrm{tr}\big(V_{\boldsymbol{z}}V^\dagger_{\boldsymbol{y}}\big)}{N_c} - \frac{1}{N_c^2}\,\frac{\mathrm{tr}\big(V_{\boldsymbol{x}}V^\dagger_{\boldsymbol{y}}\big)}{N_c}\right\rangle_Y$$

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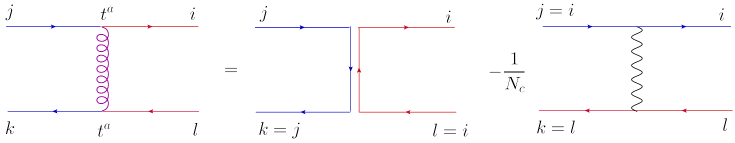

$$\left\langle \tilde{V}_{ab}(\boldsymbol{z})\, \frac{1}{N_c}\mathrm{tr}\big(V(\boldsymbol{x})t^b\, V^\dagger(\boldsymbol{y})t^a\big) \right\rangle_Y \simeq \frac{N_c}{2}\, S_Y(\boldsymbol{x},\boldsymbol{z})\, S_Y(\boldsymbol{z},\boldsymbol{y}) \quad \text{when } N_c \to \infty$$

- In the large-N_c limit, expectation values of colorless operators factorize

Fierz identities & the large N_c limit

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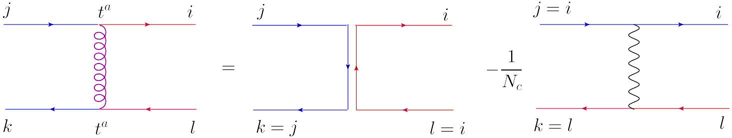


- Adding the "virtual" contribution and using $C_F = \frac{N_c^2-1}{2N_c} \simeq N_c/2$

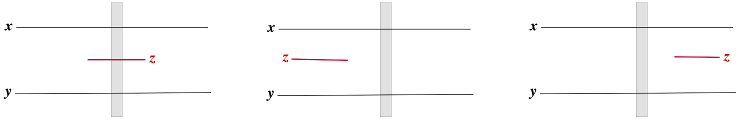
$$\left\langle \frac{\tilde{V}_{ab}(\boldsymbol{z})}{N_c}\mathrm{tr}\big(V(\boldsymbol{x})t^b\, V^\dagger(\boldsymbol{y})t^a\big) - \frac{C_F}{N_c}\,\mathrm{tr}\big(V_{\boldsymbol{x}}V^\dagger_{\boldsymbol{y}}\big)\right\rangle_Y \simeq \frac{N_c}{2}\Big\{S_Y(\boldsymbol{x},\boldsymbol{z})\, S_Y(\boldsymbol{z},\boldsymbol{y}) - S_Y(\boldsymbol{x},\boldsymbol{y})\Big\}$$

$$\frac{\partial S_Y(\boldsymbol{x},\boldsymbol{y})}{\partial Y} = \frac{\bar{\alpha}}{2\pi}\int \mathrm{d}^2\boldsymbol{z}\,\mathcal{M}_{\boldsymbol{xyz}}\big[S_Y(\boldsymbol{x},\boldsymbol{z})S_Y(\boldsymbol{z},\boldsymbol{y})-S_Y(\boldsymbol{x},\boldsymbol{y})\big]$$

- Dipole kernel: BFKL kernel in the dipole picture *(Al Mueller, 1990)*

$$\mathcal{M}_{\boldsymbol{xyz}} \equiv \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} = \left[\frac{z^i-x^i}{(\boldsymbol{z}-\boldsymbol{x})^2} - \frac{z^i-y^i}{(\boldsymbol{z}-\boldsymbol{y})^2}\right]^2; \qquad \bar{\alpha}\equiv\frac{\alpha_s N_c}{\pi}$$

- Large N_c : the original dipole splits into two new dipoles



- Differential probability for dipole splitting: $\mathrm{d}\mathcal{P} = \frac{\bar{\alpha}}{2\pi}\,\mathcal{M}_{\boldsymbol{xyz}}\,\mathrm{d}^2\boldsymbol{z}\,\mathrm{d}Y$
- Beyond large N_c one cannot write a closed equation for $S_Y(\boldsymbol{x},\boldsymbol{y})$:
  - Balitsky-JIMWLK hierarchy: n-point functions of Wilson lines

The BK equation *(Balitsky, '96; Kovchegov, '99)*

$$\frac{\partial S_Y(\boldsymbol{x},\boldsymbol{y})}{\partial Y} = \frac{\bar{\alpha}}{2\pi}\int \mathrm{d}^2\boldsymbol{z}\,\mathcal{M}_{\boldsymbol{xyz}}\big[S_Y(\boldsymbol{x},\boldsymbol{z})S_Y(\boldsymbol{z},\boldsymbol{y}) - S_Y(\boldsymbol{x},\boldsymbol{y})\big]$$

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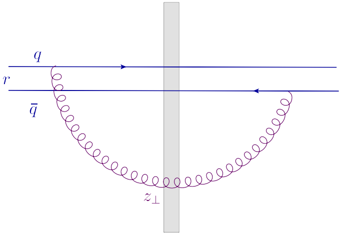
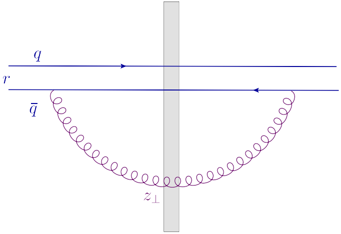
- Cancellations between large-distance contributions from "exchange" ($q\bar{q}$) and "self-energy" (qq or $\bar{q}\bar{q}$) graphs, by color neutrality





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- Cancellations between large-distance contributions from "exchange" ($q\bar{q}$) and "self-energy" (qq or $\bar{q}\bar{q}$) graphs, by color neutrality
  - color transparency: $\mathcal{M}_{\boldsymbol{xyz}} \propto (\boldsymbol{x}-\boldsymbol{y})^2 = r^2$
  - a zero-size "dipole" cannot emit, as it has zero charge
  - rapid decrease of the emission probability at large $z_\perp$:

$$\mathcal{M}_{\boldsymbol{xyz}} \simeq \frac{r^2}{(\boldsymbol{z}-\boldsymbol{x})^4} \quad \text{when } |\boldsymbol{z}-\boldsymbol{x}| \simeq |\boldsymbol{z}-\boldsymbol{y}| \gg r$$

The BK equation *(Balitsky, '96; Kovchegov, '99)*

$$\frac{\partial S_Y(\boldsymbol{x},\boldsymbol{y})}{\partial Y} = \frac{\bar{\alpha}}{2\pi}\int \mathrm{d}^2\boldsymbol{z}\,\mathcal{M}_{\boldsymbol{xyz}}\big[S_Y(\boldsymbol{x},\boldsymbol{z})S_Y(\boldsymbol{z},\boldsymbol{y}) - S_Y(\boldsymbol{x},\boldsymbol{y})\big]$$

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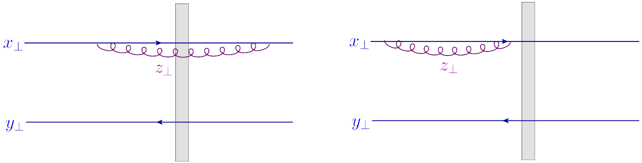
- Short-distance poles ($\boldsymbol{z}=\boldsymbol{x}$ or $\boldsymbol{z}=\boldsymbol{y}$) cancel between 'real' and 'virtual'

$$\boldsymbol{z}\to\boldsymbol{x} \implies S_Y(\boldsymbol{x},\boldsymbol{z})S_Y(\boldsymbol{z},\boldsymbol{y}) \to \mathbb{I}\times S_Y(\boldsymbol{x},\boldsymbol{y})$$



BFKL & Unitarity

- Non-linear generalization of the BFKL equation for $T_{\boldsymbol{xy}} \equiv 1 - S_{\boldsymbol{xy}}$

$$\frac{\partial T_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}}{2\pi} \int \mathrm{d}^2\boldsymbol{z}\, \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} \left[T_{\boldsymbol{xz}} + T_{\boldsymbol{zy}} - T_{\boldsymbol{xy}} - T_{\boldsymbol{xz}} T_{\boldsymbol{zy}}\right]$$

- Non-linear term T^2: the simultaneous scattering of both daughter dipoles

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$$\frac{\partial T_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}}{2\pi} \int \mathrm{d}^2\boldsymbol{z}\, \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} \left[T_{\boldsymbol{xz}} + T_{\boldsymbol{zy}} - T_{\boldsymbol{xy}}\right]$$

- When scattering is weak, $T \ll 1$, one recovers the linear BFKL equation
  - conformal symmetry: $\boldsymbol{x} \to a\boldsymbol{x} \Rrightarrow \int \mathrm{d}^2\boldsymbol{z}\mathcal{M}_{\boldsymbol{xyz}} =$ invariant
  - pure powers $r^{2\gamma}$ are eigenfunctions of the BFKL kernel:

$$\mathcal{K}_{\rm BFKL} \otimes r^{2\gamma} = \bar{\alpha}\chi(\gamma) r^{2\gamma} \qquad \text{for any } 0 < \gamma < 1$$

  - a basis of exact solutions: $T_\gamma(r, Y) \propto r^{2\gamma}\, \mathrm{e}^{\bar{\alpha}\chi(\gamma)Y}$
  - general solution: superposition in γ (Mellin transform)
  - exponential increase with Y leading to unitarity violation

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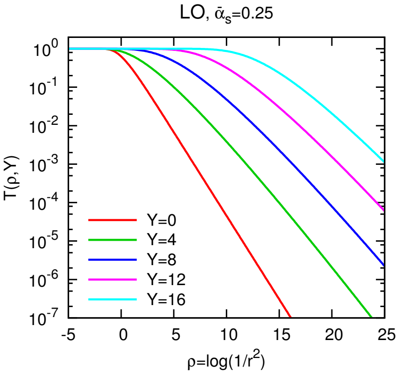
$$\frac{\partial T_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}}{2\pi} \int \mathrm{d}^2\boldsymbol{z}\, \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} \left[T_{\boldsymbol{xz}} + T_{\boldsymbol{zy}} - T_{\boldsymbol{xy}} - T_{\boldsymbol{xz}}T_{\boldsymbol{zy}}\right]$$

- The non-linear term in BK restores unitarity: $T(r, Y) \leq 1$ for any r and Y
  - $T = 0$ (no scattering) and $T = 1$ (total absorption) are fixed points
- Saturation momentum $Q_s(Y)$: $T(r, Y) = 0.5$ when $r = 1/Q_s(Y)$
  - $Q_s(Y)$ increases rapidly with Y due to the BFKL dynamics

$$T_{\text{BFKL}}\left(r = \frac{1}{Q_s}, Y\right) \sim \left(\frac{Q_0^2}{Q_s^2}\right)^{\gamma} \mathrm{e}^{\bar{\alpha}\chi(\gamma)Y} = 0.5 \quad \Longrightarrow \quad Q_s^2(Y) \simeq Q_0^2\, \mathrm{e}^{\bar{\alpha}\frac{\chi(\gamma)}{\gamma}Y}$$

  - Mellin superposition selects $\gamma_s \Rightarrow \lambda_s = \bar{\alpha}\frac{\chi(\gamma_s)}{\gamma_s}$: saturation exponent

- Numerical solutions to BK with initial condition from the MV model
- Logarithmic variable $\rho \equiv \ln(1/r^2 Q_0^2) \Longrightarrow$ large $\rho \leftrightarrow$ small r



$$T(r, Y=0) = 1 - \mathrm{e}^{-\frac{r^2 Q_0^2}{4} \ln \frac{1}{r^2 \Lambda^2}}$$

$$T(\rho, Y) = 0.5 \quad \text{when} \quad \rho = \rho_s(Y)$$

$$T(\rho, Y) \simeq \begin{cases} \mathrm{e}^{-\gamma_s(\rho-\rho_s)}\, \mathrm{e}^{-\frac{(\rho-\rho_s)^2}{2\beta_s \bar{\alpha} Y}} & (\rho > \rho_s) \\ 1 & (\rho \lesssim \rho_s) \end{cases}$$

- $\gamma_s \simeq 0.63$: anomalous dimension $1-\gamma_s \simeq 0.37$
- Geometric scaling: $T(r, Y) \simeq \left(r^2 Q_s^2(Y)\right)^{\gamma_s}$ when $\rho - \rho_s \ll \sqrt{2\beta_s \bar{\alpha} Y}$

The saturation front

- Numerical solutions to BK with initial condition from the MV model
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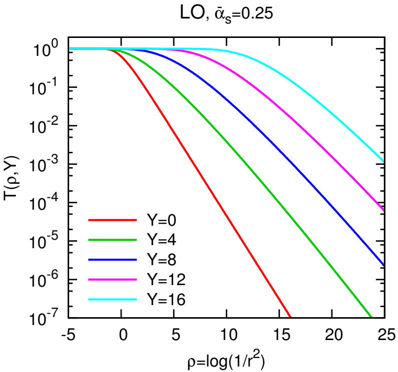
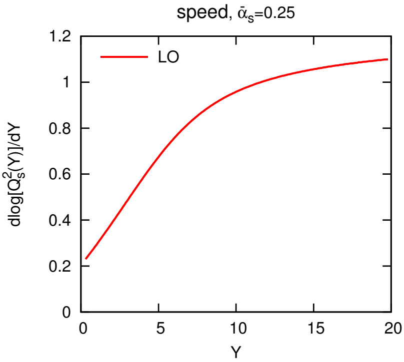
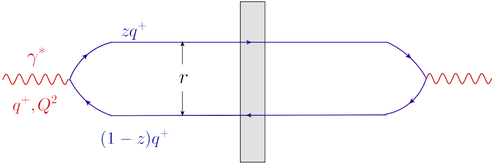




- Saturation exponent $\lambda_s \equiv \frac{\mathrm{d}\rho_s}{\mathrm{d}Y}$: the speed of the saturation front
- Constant speed for $\bar{\alpha}Y \gtrsim 4$: $\rho_s(Y) \simeq \lambda_s Y$ with $\lambda_s \simeq 4.88\bar{\alpha}$

Saturation models for HERA

- Already before BK equation: fits to small-x DIS using the idea of saturation
  - dipole factorization + saturation models for the dipole cross-section

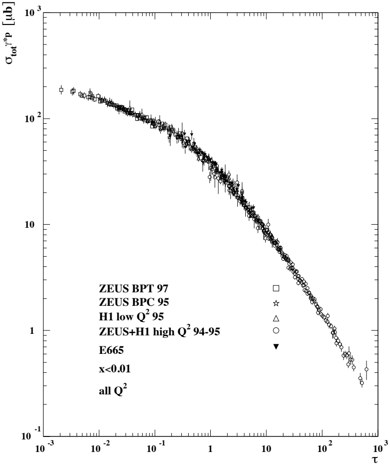


$$\sigma_{\gamma^* p}(Q^2, x) = \int_{r,z} \left|\Psi_{\gamma^*}(r, z; Q^2)\right|^2 \sigma_{\rm dip}(r, x)$$

- GBW model (Golec-Biernat, Wüsthoff, '99)

$$\sigma_{\rm dip}(r, x) = \sigma_0 \left[1 - {\rm e}^{-r^2 Q_s^2(x)}\right], \quad Q_s^2(x) \propto \frac{1}{x^\lambda}$$

- "MV model with ad-hoc evolution in Q_s"



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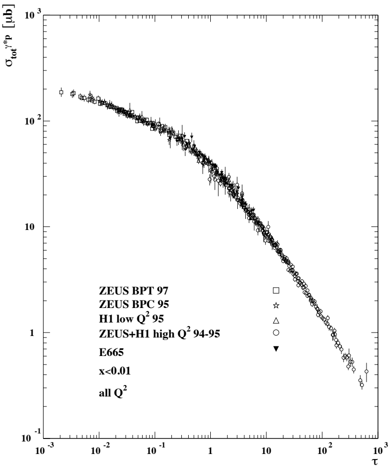
$$\sigma_{\rm dip}(r,x)=\sigma_0\Big[1-\mathrm{e}^{-r^2Q_s^2(x)}\Big],\quad Q_s^2(x)\propto\frac{1}{x^\lambda}$$

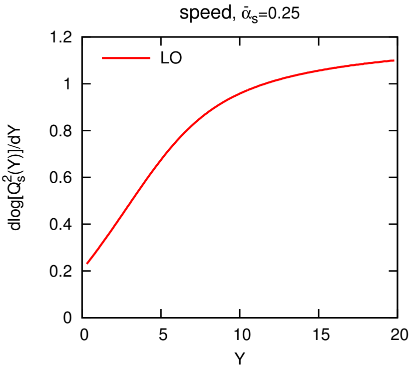
- built-in geometric scaling
- good fit at $x\le 0.01$ despite simplicity
- data clearly prefer a small value for the saturation exponent: $\lambda\simeq 0.3$
- inspired the search for geometric scaling *(Staśto, Golec-Biernat, Kwieciński, 2000)*

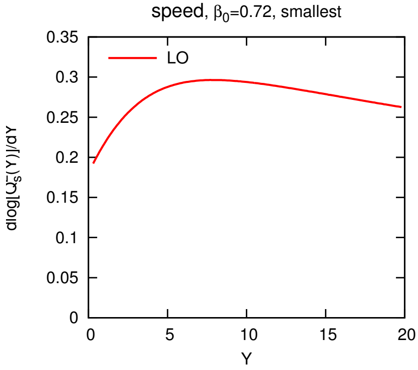
$\sigma(x,Q^2)$ vs. $\tau\equiv Q^2/Q_s^2(x)$



Adding running coupling: rcBK

- BK naturally explains geometric scaling, but $\lambda_s \simeq 4.88\bar{\alpha} \sim 1$ is way too large
- Using a running coupling dramatically slows down the evolution
    - $\lambda_s \simeq 0.3$ in good agreement with the data





- Rather successful phenomenology based on rcBK (see below)

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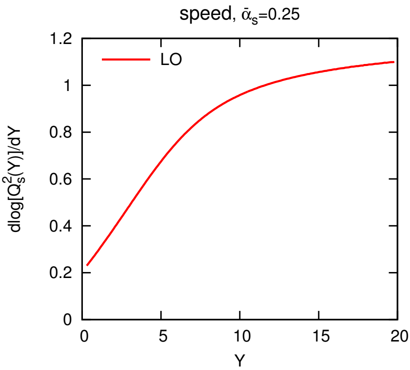
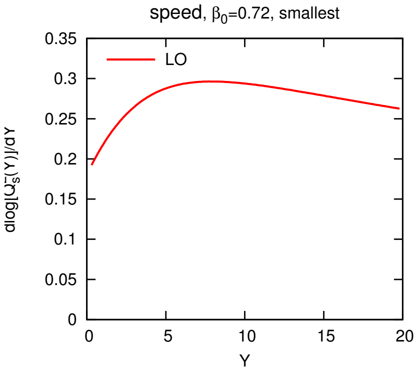
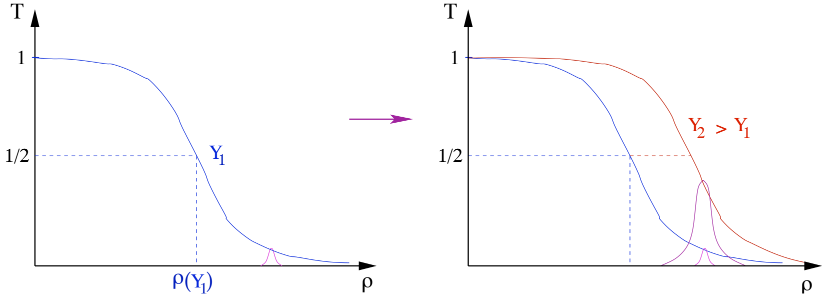




- But why should the effect of the running be so important ?!
  - the running is a next-to-leading order effect and is only logarithmic

Pulled front

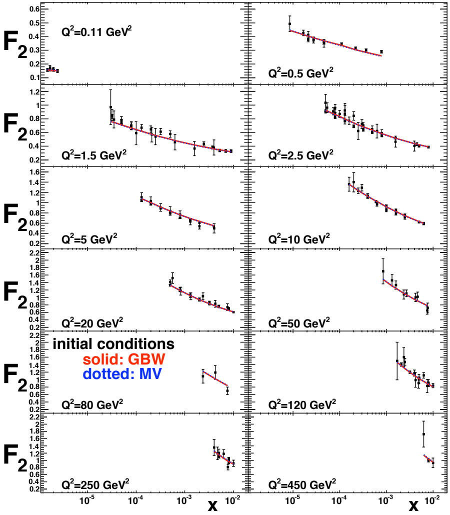
- The saturation front is pulled by the BFKL growth in the dilute tail
  - this is why one can compute λ_s from BFKL + saturation boundary
  - deep connexion to "reaction-diffusion problem" in statistical physics *(Munier and Peschanski, 2003; Iancu, Mueller and Munier, 2004)*
- The scale for the running coupling is Q_s and increases exponentially with $\bar{\alpha}Y$



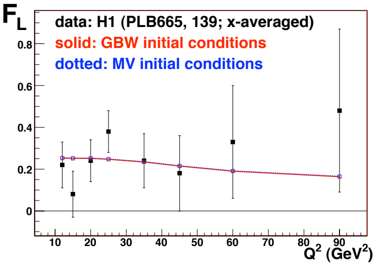
$$\alpha_s(Q_s^2) = \frac{1}{\beta_0 \ln \frac{Q_s^2}{\Lambda^2}} = \frac{1}{\beta_0(\rho_s(Y) + \rho_0)} \simeq \frac{1}{\beta_0 \lambda_s Y} \quad : \quad \text{decreasing with } Y$$

rcBK fit to F_2 at HERA (+ prediction for F_L)

(Albacete et al, hep-ph/09021112)



BK + running coupling



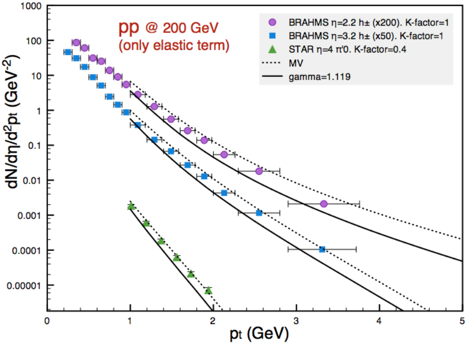
3 parameters $(R,\ Q^2_{s0},\ C)$

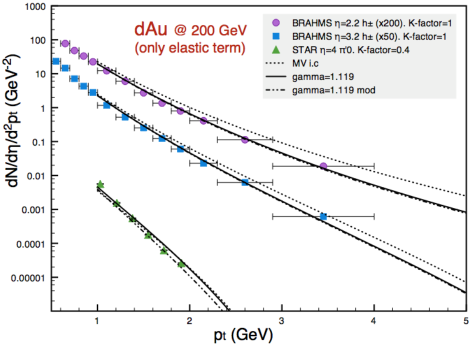
847 data points, χ^2/d.o.f. $\simeq 1$

rcBK fit to forward particle production at RHIC

(Albacete, Dumitru, Fujii, Nara, arXiv:1209:2001)

- Fit parameters: initial condition for the rcBK equation + K-factors

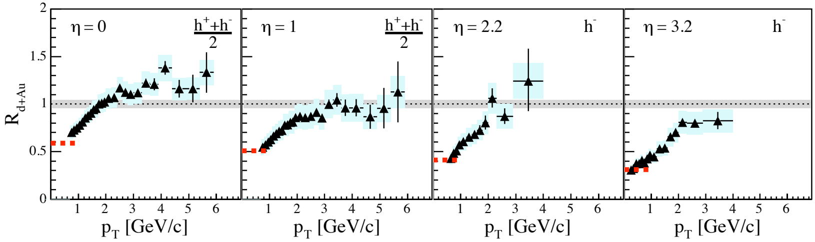




$$\left.\frac{\mathrm{d}N_h}{\mathrm{d}\eta\,\mathrm{d}^2\boldsymbol{k}}\right|_{\mathrm{LO}} = K_h \int_{x_p}^{1} \frac{\mathrm{d}z}{z^2}\,\frac{x_p}{z} q\left(\frac{x_p}{z}\right) \mathcal{S}\left(\frac{\boldsymbol{k}}{z}, X_g\right) D_{h/q}(z)$$

The nuclear modification factor

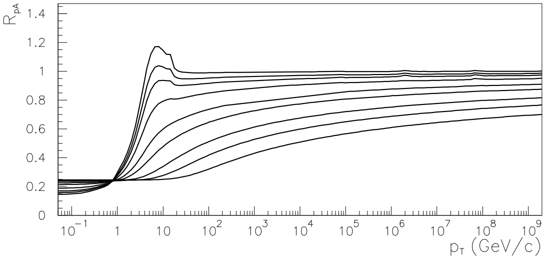
$$R_{pA} \equiv \frac{1}{A} \frac{\mathrm{d}\sigma_{pA}/\mathrm{d}^2p_\perp \mathrm{d}\eta}{\mathrm{d}\sigma_{pp}/\mathrm{d}^2p_\perp \mathrm{d}\eta}$$



- It would be equal to one if pA = incoherent superposition of pp collisions
  - any deviation from unity is a signature of nuclear (high density) effects
- At RHIC: $R_{\mathrm{d+Au}}$, hence $A \to 2A$ with $A = 197$
  - central rapidity ($\eta \simeq 0$) and $p_\perp \gtrsim 2$ GeV: $R_{\mathrm{d+Au}} > 1$ ("Cronin peak")
  - forward rapidity ($\eta > 1$) : the peak disappears when $\eta \gtrsim 1$
  - larger forward rapidities ($\eta \gtrsim 3$): $R_{\mathrm{d+Au}} < 1$ ("suppresion")

Consistent with the CGC

$$R_{pA} \equiv \frac{1}{A^{1/3}} \frac{\mathrm{d}N_{pA}/\mathrm{d}^2p_\perp \mathrm{d}\eta}{\mathrm{d}N_{pp}/\mathrm{d}^2p_\perp \mathrm{d}\eta} = \frac{1}{A^{1/3}} \frac{\mathcal{S}_A(p_\perp, X_g)}{\mathcal{S}_p(p_\perp, X_g)}$$



$$\mathcal{S}(p_\perp, X_g) = \int_{\boldsymbol{r}} \mathrm{e}^{-\mathrm{i}\boldsymbol{r}\cdot\boldsymbol{p}}\, S(\boldsymbol{r}, X_g)$$

$$X_g = \frac{p_\perp}{\sqrt{s}}\, \mathrm{e}^{-\eta}$$

$\eta =$ 0, 0.05, 0.1, 0.2, 0.4, 0.6, 1, 1.4 and 2 *(BK equation: Albacete et al, 2003)*

- Use BK equation for $S(\boldsymbol{r}, X_g)$ with initial condition from the MV model
- Exactly the same features as in the RHIC data !
- What is the underlying physical picture ?

Midrapidity: the Cronin peak

- d+Au collisions at RHIC: $\sqrt{s} = 200$ GeV, $p_\perp \sim 2$ GeV and $\eta \approx 0$
  - $x_1 = x_2 \simeq 0.01 \Longrightarrow$ little evolution, the proton is still dilute
  - nucleus: incoherent superposition of valence quarks (MV model)

$$\frac{\mathcal{S}(p_\perp)}{4\pi} \simeq \begin{cases} \dfrac{1}{Q_s^2(A)}\,\mathrm{e}^{-\frac{p_\perp^2}{Q_s^2(A)}}\,, & \text{for the nucleus} \\[2ex] \dfrac{Q_{0p}^2}{p_\perp^4}\,, & \text{for the proton} \end{cases}$$

  - remember the distinction between the two scales $Q_s^2(A)$ and Q_{0A}^2 :

$$Q_s^2(A) \,=\, Q_{0A}^2\,\ln\frac{Q_s^2(A)}{\Lambda^2}\,, \qquad Q_{0A}^2 = A^{1/3}Q_{0p}^2$$

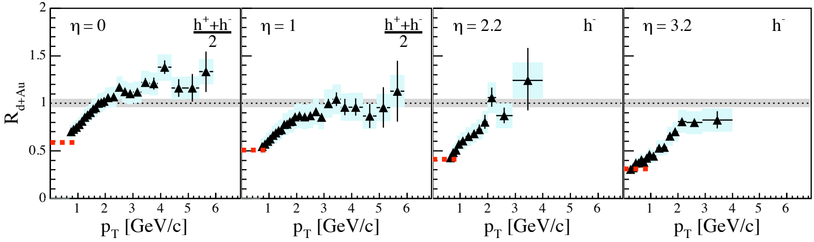
$$R_{pA} \,=\, \frac{1}{A^{1/3}}\,\frac{\mathcal{S}_A(p_\perp)}{\mathcal{S}_p(p_\perp)} \simeq \ln\frac{Q_s^2(A)}{\Lambda^2} \times \left[\frac{p_\perp^2}{Q_s^2(A)}\right]^2 \mathrm{e}^{-\frac{p_\perp^2}{Q_s^2(A)}}$$

- A peak at $p_\perp = 2Q_s(A)$ with height $\ln\,[Q_s^2(A)/\Lambda^2] > 1$

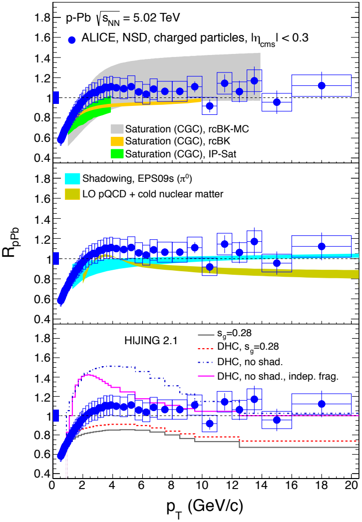
Forward rapidities: R_{pA} suppression

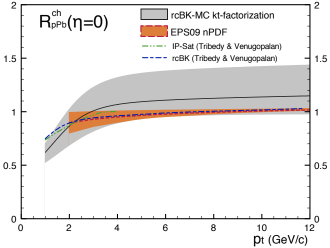
- Why is the Cronin peak washed out when increasing η (decreasing X_g) ?
- The gluon distribution in the proton rises faster than that in the nucleus
  - growth driven by BFKL dynamics in the dilute tail at $p_\perp > Q_s$
  - the logarithmic phase-space $\rho = \ln(p_\perp^2/Q_s^2)$ is larger for the proton than for the nucleus, since $Q_{0p} < Q_{0A}$

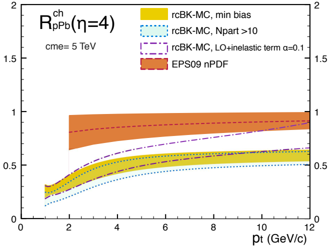
$$\rho_p = \ln \frac{p_\perp}{Q_{0p}^2} > \rho_A = \ln \frac{p_\perp}{Q_{0A}^2} \quad \text{since} \quad Q_{0A}^2 = A^{1/3} Q_{0p}^2$$



$R_{\mathrm{p+Pb}}$ at the LHC for central rapidities

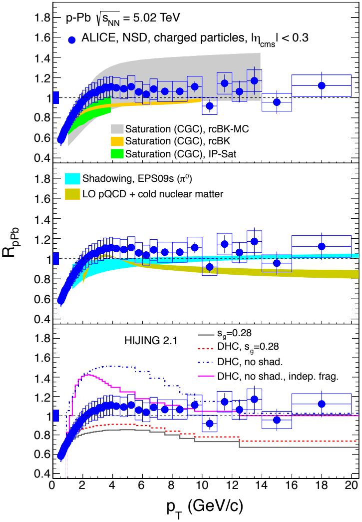


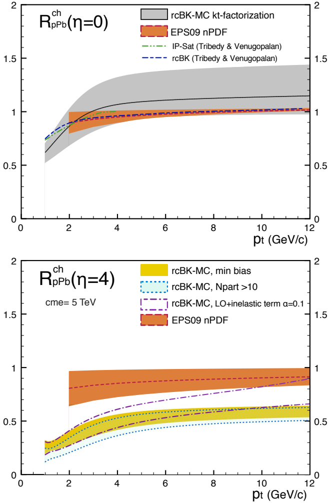




- Midrapidity ($\eta \simeq 0$) at the LHC is like $\eta \sim 2$ at RHIC: $x_1 \sim x_2 \sim 10^{-3}$
- Cronin peak and small evolution compensate each other: $R_{pA} \simeq 1$

$R_{\mathrm{p+Pb}}$ at the LHC for central rapidities

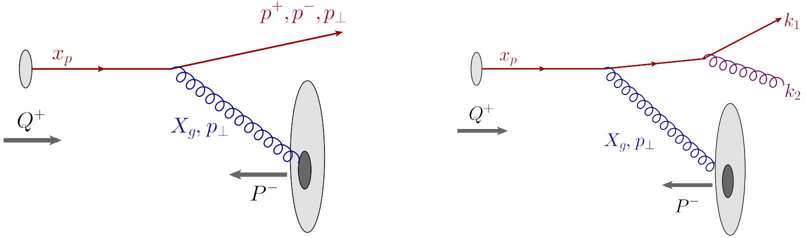




- Various models could be differentiated by going to forward rapidities
- This could be measured e.g. by LHCb (large η & semi-hard $p_\perp$)

Forward di-hadron production in pA collisions

- Saturation effects enter single-particle production via modifications in the yield and spectrum at semi-hard transverse momenta $k_\perp \sim Q_s(X_g)$
- In 2-particle production, they also affect the angular correlations



$$x_p = \frac{k_{1\perp}}{\sqrt{s}}\,\mathrm{e}^{\eta_1} + \frac{k_{2\perp}}{\sqrt{s}}\,\mathrm{e}^{\eta_2}\,, \qquad X_g = \frac{k_{1\perp}}{\sqrt{s}}\,\mathrm{e}^{-\eta_1} + \frac{k_{2\perp}}{\sqrt{s}}\,\mathrm{e}^{-\eta_2}$$

- Focus on forward rapidities for both particles $\Longrightarrow$ $x_p \sim \mathcal{O}(1)$ and $X_g \ll 1$

Forward di-hadron production in pA collisions

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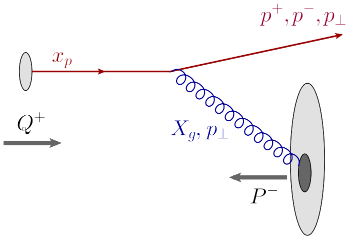
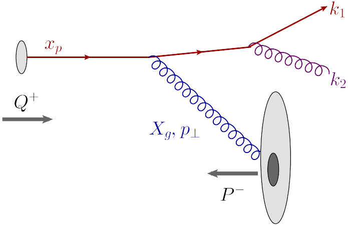


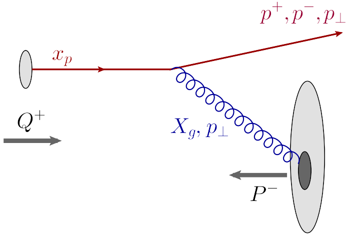


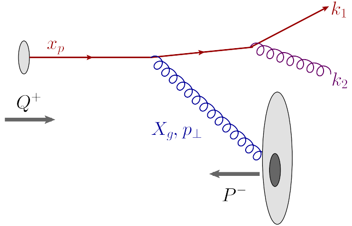
- Measure pairs of particles and extract their correlation in azimuthal angle $\Delta\phi = \phi_2 - \phi_1$

$$\mathcal{C}(\Delta\phi) \equiv \frac{\mathrm{d}N_{\text{pair}}}{\mathrm{d}^2p_{1\perp}\mathrm{d}\eta_1\mathrm{d}^2p_{2\perp}\mathrm{d}\eta_2} - \frac{\mathrm{d}N}{\mathrm{d}^2p_{1\perp}\mathrm{d}\eta_1}\,\frac{\mathrm{d}N}{\mathrm{d}^2p_{2\perp}\mathrm{d}\eta_2}$$

Forward di-hadron production in pA collisions

- Saturation effects enter single-particle production via modifications in the yield and spectrum at semi-hard transverse momenta $k_\perp \sim Q_s(X_g)$
- In 2-particle production, they also affect the angular correlations

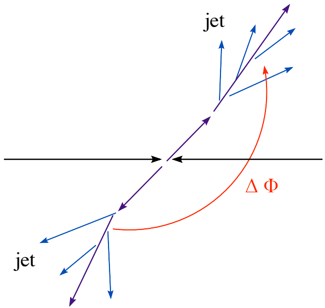


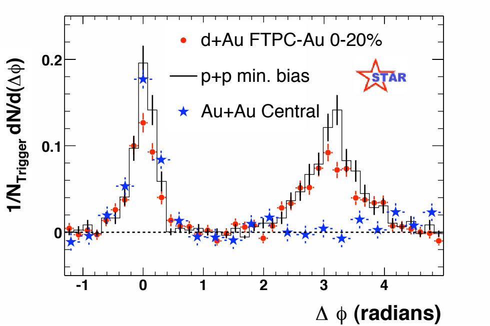


- Collinear factorization : $\boldsymbol{k}_{1\perp} + \boldsymbol{k}_{2\perp} \simeq 0 \Longrightarrow$ a peak at $\Delta\phi = \pi$
  - a pair of hadrons propagating back-to-back in the transverse plane
- In the presence of gluon saturation: $|\boldsymbol{k}_{1\perp} + \boldsymbol{k}_{2\perp}| \simeq Q_s(X_g)$
  - a broadening $\delta\phi \sim Q_s/k_\perp$ of the peak at $\Delta\phi = \pi$

Di–hadron azimuthal correlations at RHIC

- The reality is more nuanced
  - even in pp collisions, there is significant broadening in $\Delta\phi$, due to recoil in jet fragmentation
  - in pA or AA, high-density effects may also reflect final-state interactions, and not just gluon saturation

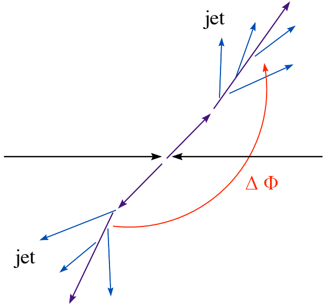


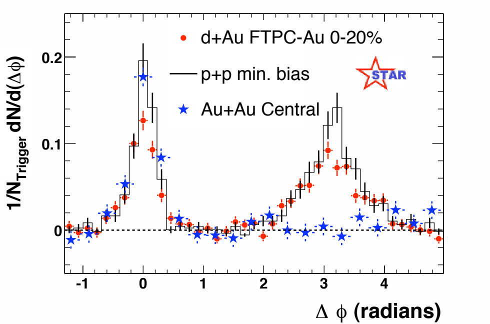


- p+p, d+Au and Au+Au collisions at RHIC (STAR)

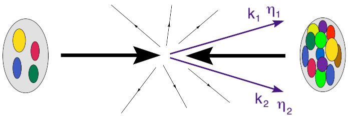
Di–hadron azimuthal correlations at RHIC

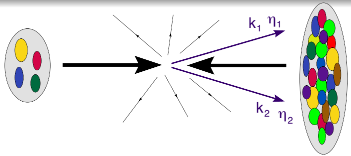
- Midrapidities $(\eta_1 \sim \eta_2 \simeq 0)$ and semi-hard $p_\perp \sim 1 \div 3$ GeV
  - p+p or d+Au: the peak at $\Delta\Phi = \pi$ is visible and equally pronounced
  - Au+Au : strong suppression of the 'away peak' (final state effect)
- Broadening in d+Au is controlled by jet fragmentation, like in p+p
- What happens if one moves to forward rapidities (larger $Q_s(A)$) ?

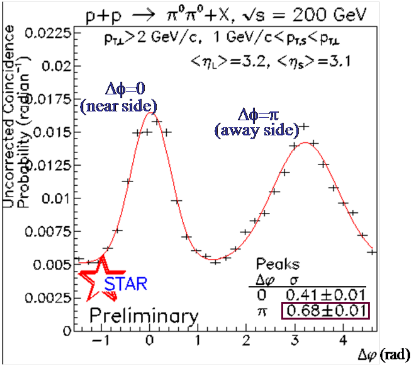


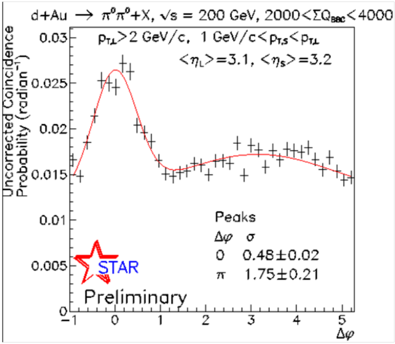


Forward rapidities: p+p vs. d+Au





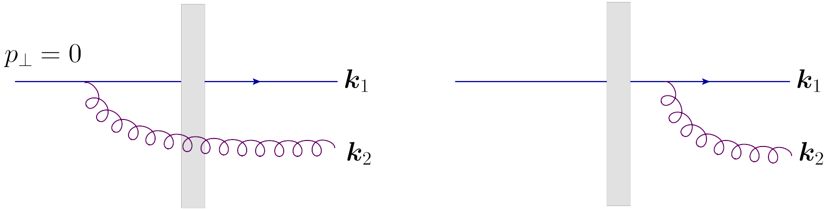




- The broadening in d+Au is considerably stronger
- Predicted by the CGC *(Marquet, 2007; Albacete and Marquet, 2010)*

2 particle production in the CGC

- The collinear quark radiates a gluon prior to, or after, the scattering



- Up to four Wilson lines in the cross–section
- At large N_c, this factorizes into color dipoles and quadrupoles

$$\langle Q_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4}\rangle_Y = \frac{1}{N_c}\langle \mathrm{tr}(V^\dagger_{\boldsymbol{x}_1} V_{\boldsymbol{x}_2} V^\dagger_{\boldsymbol{x}_3} V_{\boldsymbol{x}_4})\rangle_Y$$

  - a generalization of the Weiszäcker-Williams gluon TMD *(Cédric Lorcé)*

- This property holds for any multi-particle final state at large N_c *(Kovner and Lublinsky, 2012; Dominguez, Marquet, Stasto, and Xiao, '12)*

With due respect to the target

- How to compute the quadrupole ? How to work at $N_c = 3$?
- Return to the viewpoint of target evolution: the CGC target average:

$$\langle Q_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4}\rangle_Y = \int [\mathcal{D}\rho]\; W_Y[\rho]\; \frac{1}{N_c}\,\mathrm{tr}(V^\dagger_{\boldsymbol{x}_1} V_{\boldsymbol{x}_2} V^\dagger_{\boldsymbol{x}_3} V_{\boldsymbol{x}_4})$$

- The Wilson lines involve the A^- component of the color field in the target
- $A^-_a(x)$: the classical color field produced by color charges with density ρ^a

$$D^{ab}_\nu F^{\nu\mu}_b(x) = J^\mu_a(x) = \delta^{\mu-}\rho_a(x^+, \boldsymbol{x}) \implies -\nabla^2_\perp A^-_a(x) = \rho_a(x^+, \boldsymbol{x})$$

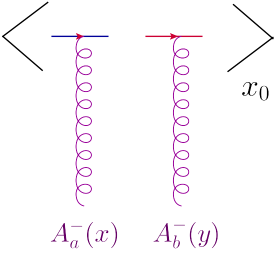
  - $J^\mu_a \propto v^\mu = \delta^{\mu-}$ for a ultrarelativistic left-mover
  - light-cone gauge $A^+_a = 0$ (convenient for collision with a right mover)
  - Yang-Mills equations linearize since $A^\mu_a = \delta^{\mu-} A^-_a$ for this special J^μ

- The functional probability distribution $W_Y[\rho]$ describe the correlations of the color charges with $x' \gg x$, where $Y = \ln(1/x)$.

JIMWLK evolution

(Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, and Kovner, 97–00)

- The relevant color charges at small-x (leading logarithmic approximation):
  - valence quarks + soft gluons with $1 \gg x' \gg x$
- $W_Y[\rho]$ is built by integrating out soft gluon fluctuations in (small) layers of x
  - $x' \to bx'$ with $b \ll 1$ but such that $\bar{\alpha}\ln(1/b) \ll 1$ as well
- Initial condition at low energy ($x_0 \sim 0.01$): MV model (valence quarks)



- independent color sources
- Gaussian weight function

$$W_0[\rho] = \mathcal{N} \exp\left\{ -\int_{x^+,\boldsymbol{x}} \frac{\rho_a(x)\rho_a(x)}{\mu^2(x)} \right\}$$

- $\mu^2(x)$: density of color charge squared

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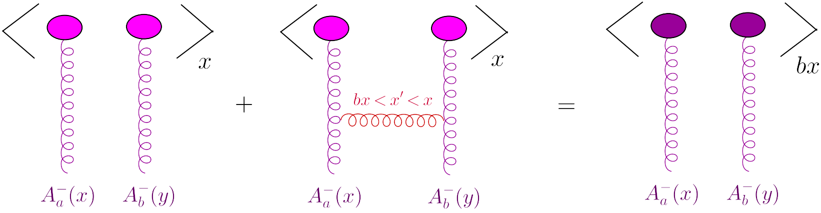
- One step in the quantum evolution $\Longrightarrow$ JIMWLK Hamiltonian



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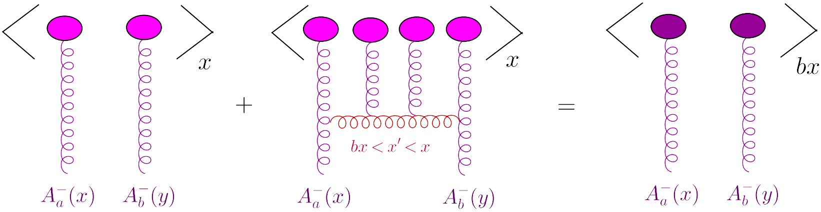

- The quantum gluon can scatter of the strong color fields generated in previous steps $\Longrightarrow$ non-linear evolution

JIMWLK evolution

(Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, and Kovner, 97–00)

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  - $x' \to bx'$ with $b \ll 1$ but such that $\bar{\alpha}\ln(1/b) \ll 1$ as well
- One step in the quantum evolution $\Longrightarrow$ JIMWLK Hamiltonian

$$\frac{\partial W_Y[\rho]}{\partial Y} = H_{\rm JIMWLK}\left[\rho, \frac{\delta}{\delta\rho}\right] W_Y[\rho]$$

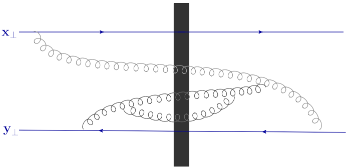
- BK (Balitsky) equations are obtained after an integration by parts:

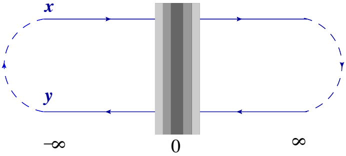
$$\frac{\partial}{\partial Y}\langle\hat{S}\rangle_Y = \int[\mathcal{D}\rho]\left(HW_Y[\rho]\right)\hat{S}[\rho] = \int[\mathcal{D}\rho]\, W_Y[\rho]\left(H\hat{S}[\rho]\right) = \left\langle H\hat{S}\right\rangle_Y$$

- But JIMWLK equation can actually be solved numerically (for $N_c = 3$)

JIMWLK evolution in Langevin form (1)

- Useful to compare projectile (dipole) and target (nucleus) evolutions

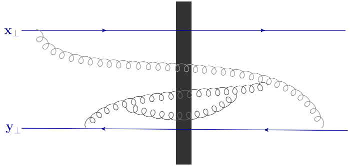


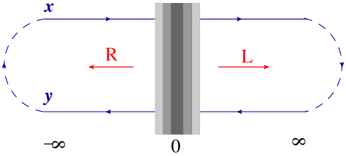


  - projectile: gluon emissions closer and closer to the target
  - target: color charges further and further away from the valence quarks
- Uncertainty principle: decreasing $x = k^-/P^- \leftrightarrow$ increasing $\Delta x^+ \sim 1/k^-$
- JIMWLK evolution builds the color charge distribution in layers of x^+
- New sources are one-loop quantum fluctuations
  - random variables with a Gaussian distribution
  - can equivalently be represented as a Gaussian noise
- A Langevin equation: random walk in the space of the Wilson lines

JIMWLK in Langevin form *(Blaizot, E.I., Weigert, '03)*

- Discretize the rapidity interval: $Y = n\epsilon$, $\epsilon \equiv \ln(1/b)$





$$V_{\boldsymbol{x}}(n\varepsilon+\varepsilon) \;=\; \exp\big(i\varepsilon\alpha^a_{L\boldsymbol{x}}t^a\big)\, V_{\boldsymbol{x}}(n\varepsilon)\; \exp\big(-i\varepsilon\alpha^b_{R\boldsymbol{x}}t^b\big)$$

- $\alpha^a_{R,L}$: the change δA^-_a at larger negative (R) or positive (L) values of x^+

$$\alpha^a_{L\boldsymbol{x}} \;=\; g\int_{\boldsymbol{z}} \frac{x^i-z^i}{(\boldsymbol{x}-\boldsymbol{z})^2}\,\nu^{ia}_{\boldsymbol{z}}\,, \qquad \alpha^a_{R\boldsymbol{x}} \;=\; g\int_{\boldsymbol{z}} \frac{x^i-z^i}{(\boldsymbol{x}-\boldsymbol{z})^2}\,\tilde{V}^{ab}_{\boldsymbol{z}}\,\nu^{ib}_{\boldsymbol{z}}$$

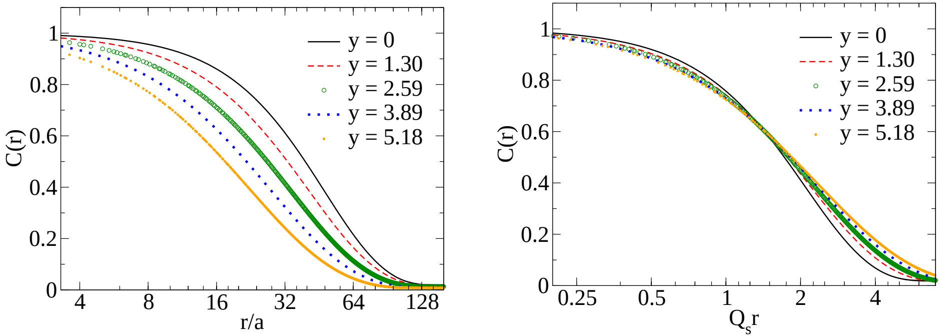
- Noise ν^a: random color charge of the newly emitted gluon

$$\langle \nu^{ia}_{\boldsymbol{x}}(m\varepsilon)\nu^{jb}_{\boldsymbol{y}}(n\varepsilon)\rangle = \frac{1}{\varepsilon}\,\delta_{mn}\delta^{ij}\delta^{ab}\delta_{\boldsymbol{xy}}$$

- Well suited for numerics: 2D lattice *(Weigert and Rummukainen, '03)*

Solving JIMWLK via Langevin

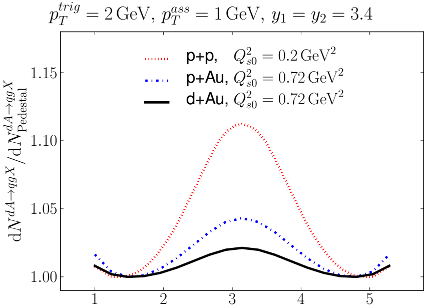
- Several numerical implementations *(Weigert and Rummukainen, '03) Lappi (2011); Schenke et al (since 2012); Roiesnel (2016)*
- Here: the lattice calculation of the dipole S-matrix par T. Lappi (2011)

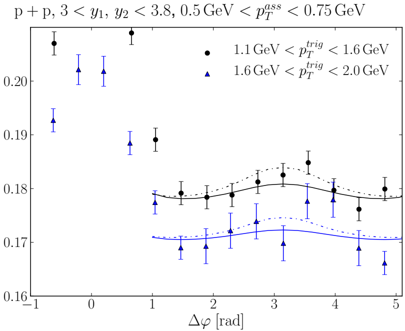


- $C(r) \equiv S(r, Y)$ as a function of r and of $rQ_s(Y) \Longrightarrow$ geometric scaling

The mean field approximation

- Gaussian Ansatz for $W_Y[\rho]$: "MV model with Y-dependent 2-point function"
  - all Wilson lines correlators (quadrupole etc) can be related to the dipole S-matrix, as obtained by solving the BK equation



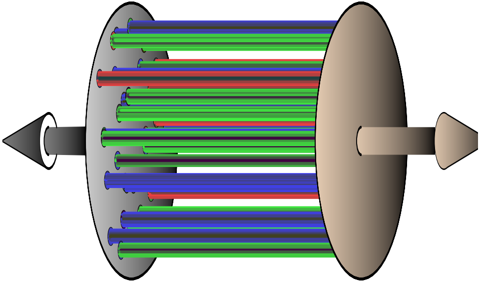


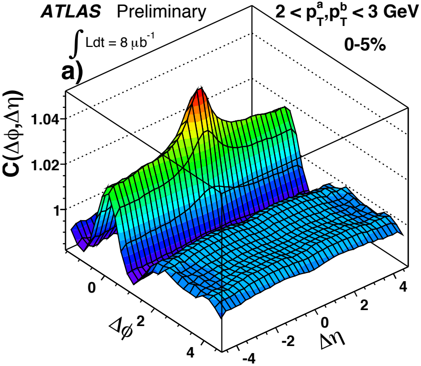
- Left: different combinations projectile–target

  (Lappi and Mäntysaari, 2012; see also Stasto, Xiao, Yuan, 2011)

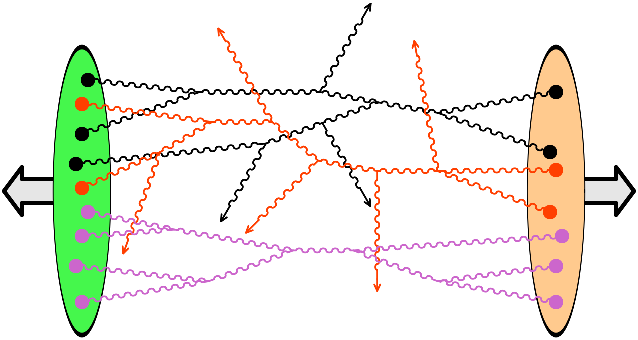
- Right: comparison with RHIC data (PHENIX, 2012)

AA collisions : Glasma & the Ridge



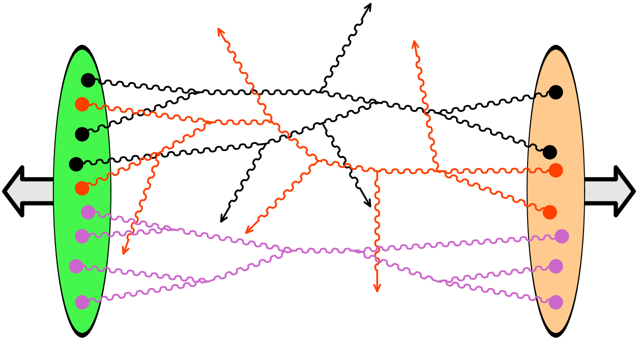


Nucleus–nucleus collisions



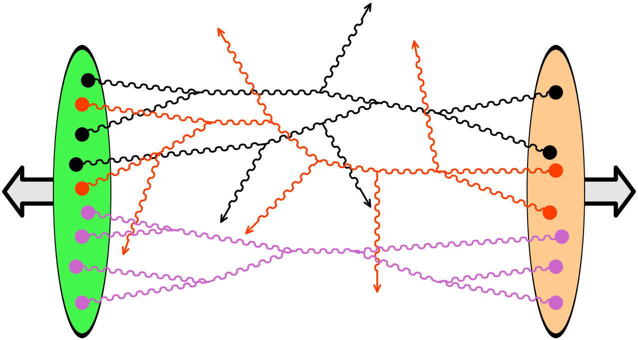
- "Dense–dense scattering" : much more complicated !
- Non–linear effects enter at all levels
  - in both incoming wavefunctions: gluon saturation
  - in the scattering process: multiple interactions
  - in the partonic medium created by the early scattering: final–state interactions

Nucleus–nucleus collisions



- "Dense–dense scattering" : much more complicated !
- Non–linear effects enter at all levels
  - 2 CGC weight functions: $W_{Y_1}[\rho_1]$, $W_{Y_2}[\rho_2]$
  - in the scattering process: multiple interactions
  - in the partonic medium created by the early scattering: final–state interactions

Nucleus–nucleus collisions



- "Dense–dense scattering" : much more complicated !
- Non–linear effects enter at all levels
  - 2 CGC weight functions: $W_{Y_1}[\rho_1]$, $W_{Y_2}[\rho_2]$
  - classical Yang–Mills equations with 2 sources: $\rho_1,\ \rho_2$
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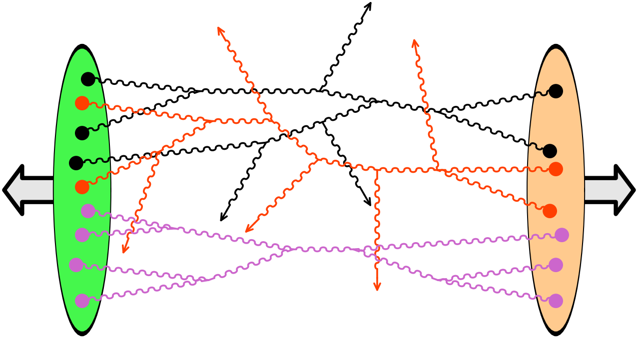

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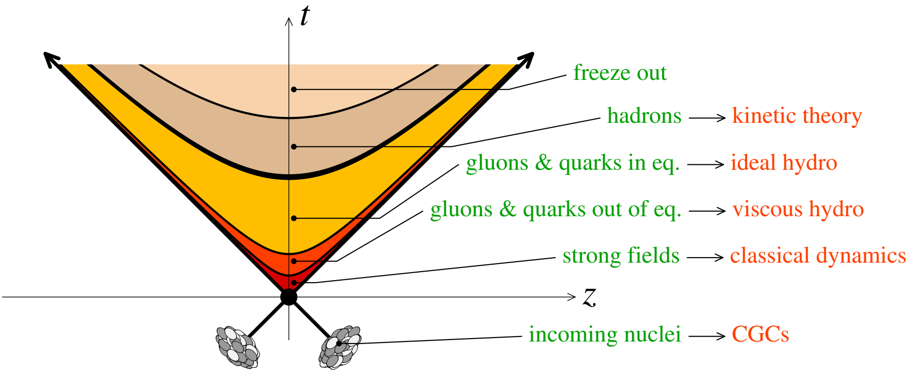
  - kinetic theory, hydrodynamics, quark-gluon plasma, ...

The initial conditions for heavy ion collisions



- The CGC describes particle production at early times: Glasma
- Initial conditions for the subsequent evolution of this partonic matter
- The state of the partonic matter at proper time $\tau \sim 1/Q_s$

CGC factorization for AA collisions

- Numerically solve classical YM equations with 2 sources (2D lattice)

$$D_\nu F^{\nu\mu}(x) = \delta^{\mu +}\rho_1(x) + \delta^{\mu -}\rho_2(x)$$

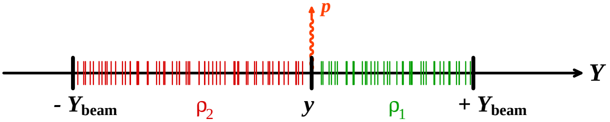
- Decompose the classical field A_a^μ in Fourier modes

  ▷ gluon spectrum for given configurations ρ_1 and ρ_2 (“event-by-event”)

- Average over ρ_1 and ρ_2 using the CGC distributions of the nuclei

$$\left\langle \frac{\mathrm{d}N}{\mathrm{d}Y\,\mathrm{d}^2p_\perp} \right\rangle = \int [\mathcal{D}\rho_1 \mathcal{D}\rho_2]\, W_{Y_{\mathrm{beam}}-Y}[\rho_1]\, W_{Y_{\mathrm{beam}}+Y}[\rho_2]\, \frac{\mathrm{d}N}{\mathrm{d}Y\,\mathrm{d}^2p_\perp}\bigg|_{\mathrm{class}}$$

  ▷ JIMWLK evolution from Y_{beam} up to the rapidity Y of the produced gluon



The color field of a single nucleus

- What are the chromo-electric and magnetic fields created by a ultrarelativistic slab of charges ?
  - non-Abelian generalization of the Liénard-Wiechert potentials
  - Weiszäcker-Williams fields describing quasi-real photons/gluons

$$D_\nu^{ab} F_b^{\nu\mu}(x) = \delta^{\mu -} \rho_a(x^+, \boldsymbol{x}) \implies A_a^\mu = \delta^{\mu -} A_a^-(x^+, \boldsymbol{x})$$

- Just one component (A^-), independent of x^- (LC time for the left mover)
  - field commutators vanish: $F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu$, as in QED

$$F_a^{i+} = 0, \quad F_a^{+-} = \partial^+ A_a^- = \frac{\partial A_a^-}{\partial x^-} = 0\,, \quad F_a^{i-} = \partial^i A_a^- \ : \ \text{non-zero}$$

- Non-zero transverse components for the electric and the magnetic fields

$$F^{i\pm} = \frac{1}{\sqrt{2}}\big(F^{i0} \pm F^{i3}\big) \implies \begin{cases} E^1 \equiv -F^{10} = F^{13} \equiv B^2 \\ E^2 \equiv -F^{20} = F^{23} \equiv -B^1 \end{cases}$$

$$F^{+-} = 0 \ \Rightarrow \ E^3 = B^3 = 0$$

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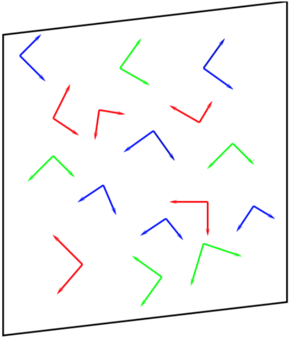


$$\boldsymbol{E}_a \perp \boldsymbol{B}_a \perp \boldsymbol{z}$$

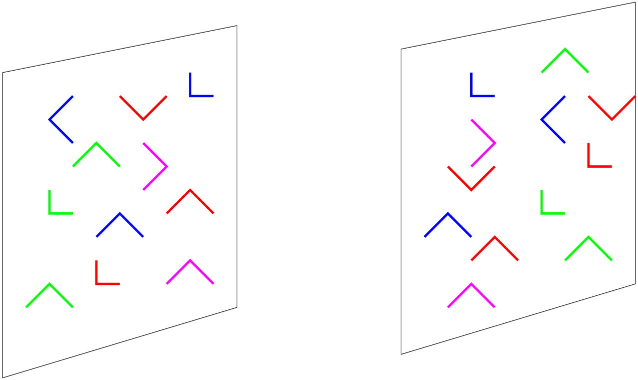
$$\boldsymbol{E}_\perp \cdot \boldsymbol{B}_\perp = 0\,, \quad |\boldsymbol{E}_\perp| = |\boldsymbol{B}_\perp| \sim \frac{1}{g}$$

- transverse polarizations
- chromo-electromagnetic waves
- Lorentz contraction: $\propto \delta(x^+)$

- Fields vary over a distance $\sim 1/Q_s \Longrightarrow$ gluons typically have $k_\perp \sim Q_s$
- Fields have strength $\sim 1/g \Longrightarrow$ gluons have occupation numbers $\sim 1/\alpha_s$

The scattering between two color sheets

- Prior to scattering: purely transverse fields, localized near the 2 light-cones



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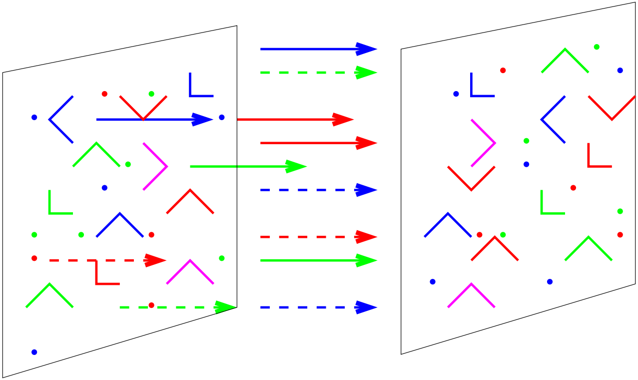
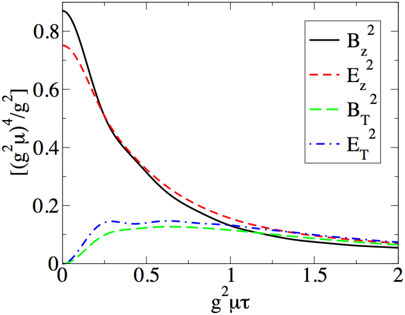
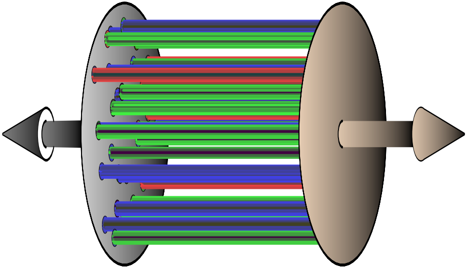


- During collision, mutual color rotations induce color charges on the sheets
  - longitudinal chromo-electric and chromo-magnetic fields
  - color strings (flux tubes) with typical transverse size $1/Q_s$

Glasma *(McLerran and Lappi, 06)*

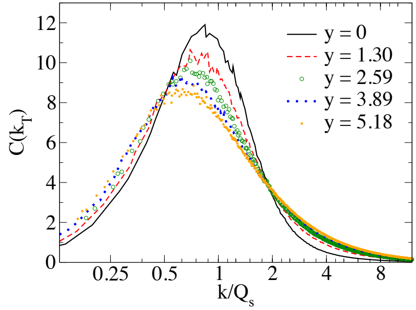
- The 'valence' charges of the 2 nuclei rapidly separate from each other
- The color field between the 2 recessing nuclei: 'glasma' ('glass' + 'plasma')

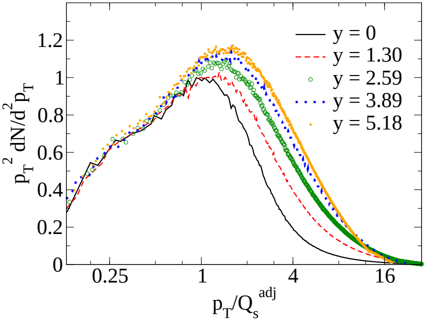




- After a time $\tau \sim 1/Q_s$, the transverse fields are regenerated
- By that time, the partonic system becomes dilute (field strengths become of $\mathcal{O}(1)$), due to longitudinal expansion
- Fourier mode decomposition $\Longrightarrow$ gluon production

- Numerical solutions to classical YM equations + rcJIMWLK evolution





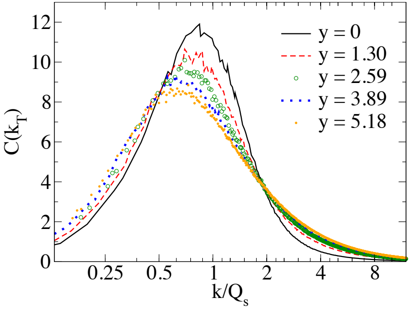
- Left: unintegrated gluon distribution for any of the two incoming nuclei

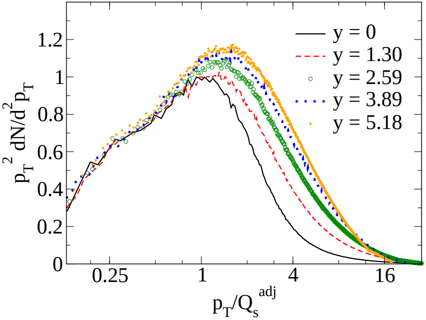
$$C(k_\perp, Y) \equiv k_\perp^2 \mathcal{S}(k_\perp, Y) = k_\perp^2 \int_{\boldsymbol{r}} \mathrm{e}^{-\mathrm{i}\boldsymbol{r}\cdot\boldsymbol{k}}\, S(\boldsymbol{r}, Y)$$

  - MV model (no evolution): $C(k_\perp) \simeq Q_0^2/k_\perp^2$

The gluon spectrum at early times *(T. Lappi, 2011)*

- Numerical solutions to classical YM equations + rcJIMWLK evolution





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  - BK evolution $\Rightarrow$ anomalous dimension: $C(k_\perp, Y) \simeq \left[Q_s^2(Y)/k_\perp^2\right]^{\gamma_s}$

The gluon spectrum at early times *(T. Lappi, 2011)*

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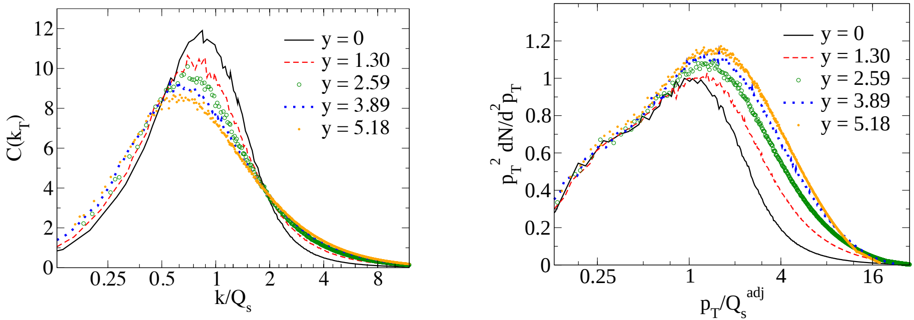
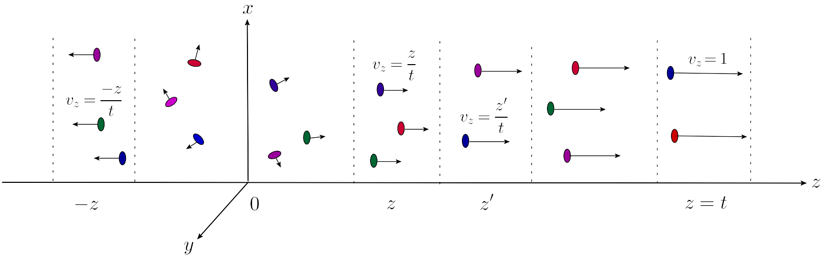


- Right: spectrum of gluons produced in AA collisions at time $\tau \sim 1/Q_s(Y)$
  - high $p_\perp \gg Q_s(Y)$: the same anomalous dimension as in $C(k_\perp, Y)$
  - small $p_\perp \lesssim Q_s(Y)$: universal shape (non-linear effects in classical YM)
- Not an observable though: spectrum is modified by final state interactions
- Correlations in multi-particle production have more chances to survive

Longitudinal expansion

- Gluons liberated by the collision at $t = 0$ have transverse momenta $k_\perp \sim Q_s$ and longitudinal momenta $|k_z| \sim x_1 P_1^+$ or $|k_z| \sim x_2 P_2^-$
- After a time $t \sim 1/Q_s$, they separate from each other along the z axis
  - particles which at time t are located at z have a velocity $v_z = \frac{z}{t}$



- Particles with different velocities $v_z \simeq k_z/Q_s$ can interact with each other only at early times $t \lesssim 1/Q_s$
- In any slice of z, the distribution in the transverse plane is roughly isotropic

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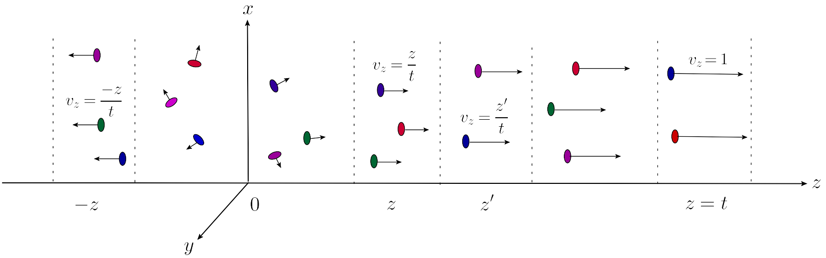
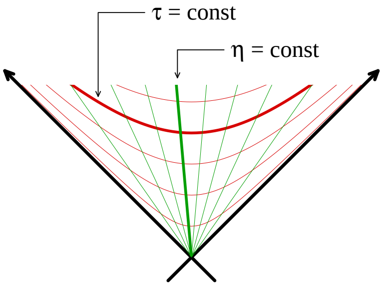


- Bjorken (1982): Particle distribution is independent of v_z: boost-invariant
- Natural in perturbative QCD (at least, in the classical approximation)

Boost invariance

- The classical field (& particle production at early times) is boost invariant

  ▷ it depends upon proper time τ but not upon space–time rapidity η_s



$$\tau \equiv \sqrt{t^2 - z^2} = \sqrt{2x^+x^-}$$

$$\eta_s \equiv \frac{1}{2}\ln\frac{t+z}{t-z} = \frac{1}{2}\ln\frac{x^+}{x^-}$$

- Under a boost with velocity β

$$\eta_s \longrightarrow \eta_s + \tanh\beta$$

- For a particle produced at $\tau = 0$ and undergoing free streaming $(z = v_z t)$, η_s coincides with the momentum-space pseudo-rapidity η:

$$\eta_s = \frac{1}{2}\ln\frac{1+v_z}{1-v_z} = \frac{1}{2}\ln\frac{p+p_z}{p-p_z} = -\ln\tan\frac{\theta}{2} = \eta$$

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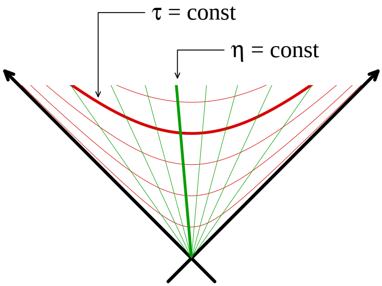


$$\tau \equiv \sqrt{t^2 - z^2} = \sqrt{2x^+x^-}$$

$$\eta_s \equiv \frac{1}{2}\ln\frac{t+z}{t-z} = \frac{1}{2}\ln\frac{x^+}{x^-}$$

$$y = \frac{1}{2}\ln\frac{p^+}{p^-} = \frac{1}{2}\ln\frac{2(p^+)^2}{p_\perp^2} \simeq \ln\frac{xP^+}{p_\perp}$$

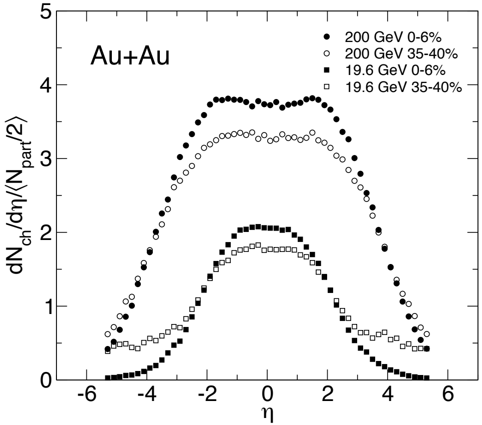
- For ultrarelativistic particles ($E \simeq p$), it also coincides with the usual momentum-space rapidity y:

$$\eta_s = \frac{1}{2}\ln\frac{1+v_z}{1-v_z} = \frac{1}{2}\ln\frac{E+p_z}{E-p_z} = y = y_{\rm beam} - \ln\frac{1}{x}$$

Multiplicity: the Feynman plateau

- Bremsstrahlung $\Longrightarrow$ the gluon distribution $xG^{(0)}(x, Q^2)$ at leading order (the classical approximation) is independent of x, hence of $y \simeq \eta \simeq \eta_s$

  ▷ the x-dependence comes fully from the quantum evolution



- $\mathrm{d}N_{\mathrm{ch}}/\mathrm{d}\eta$ as a function of η (RHIC, PHOBOS): flat at $|\eta| \le \eta_{\mathrm{beam}}$

Rapidity correlations

- Consider pairs of particles in the final state with different rapidities

  ▷ hadrons which propagate at different angles w.r.t. the collision axis

- Construct the 2-hadron correlation in η and $\boldsymbol{p}_\perp$:

$$\mathcal{C}(\Delta\phi, \Delta\eta) \equiv \frac{\mathrm{d}N_{\mathrm{pair}}}{\mathrm{d}^2p_{1\perp}\mathrm{d}\eta_1\mathrm{d}^2p_{2\perp}\mathrm{d}\eta_2} - \frac{\mathrm{d}N}{\mathrm{d}^2p_{1\perp}\mathrm{d}\eta_1}\,\frac{\mathrm{d}N}{\mathrm{d}^2p_{2\perp}\mathrm{d}\eta_2}$$

  ▷ mostly interested in the distribution in $\Delta\eta = \eta_1 - \eta_2$ and in $\Delta\phi = \phi_1 - \phi_2$

- This correlation is built at early stages: $\tau \lesssim 1/Q_s \simeq 0.2\,\mathrm{fm}$ if $Q_s = 1\,\mathrm{GeV}$

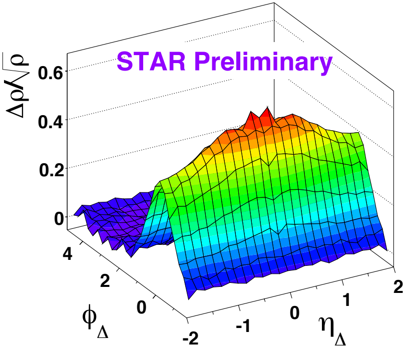
  ▷ it teaches us about the initial conditions/glasma

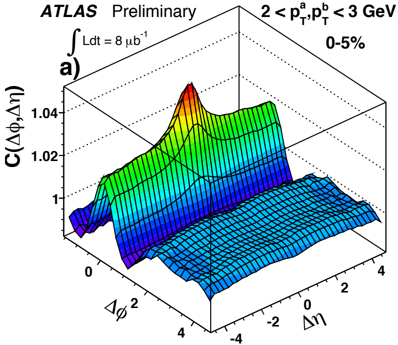
- To be correlated via the production mechanism, 2 partons must originate from a same interaction region with transverse area $\sim 1/Q_s^2$

- What would CGC / boost-invariant longitudinal expansion predict ?

  ▷ a Feynman plateau in $\Delta\eta$, at least up to $\Delta\eta \sim 1/\bar{\alpha}$

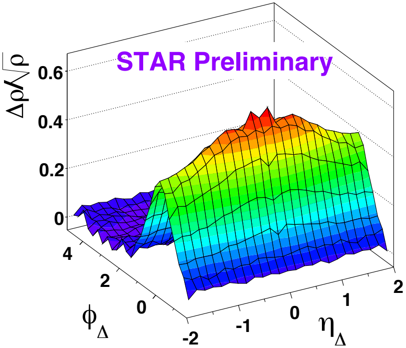
- "The ridge": di-hadron correlations long-ranged in $\Delta\eta$ & narrow in $\Delta\phi$
- Abundantly observed in AA collisions at both RHIC and the LHC
- A "trivial" peak around $\Delta\eta = 0$ and $\Delta\phi = 0$
  - pairs of particles belonging to a same jet (and there are many !)

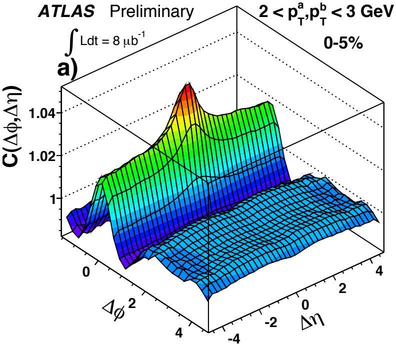




The Ridge in AA collisions

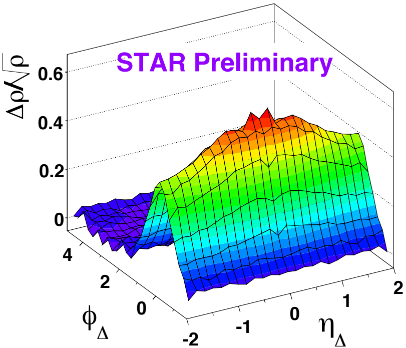
- "The ridge": di-hadron correlations long-ranged in $\Delta\eta$ & narrow in $\Delta\phi$
- Abundantly observed in AA collisions at both RHIC and the LHC
- A plateau in rapidity extending over an interval as large as $\Delta\eta = 8$
  - boost invariance for particles created in the same interaction region

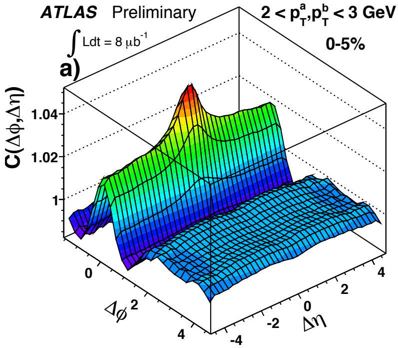




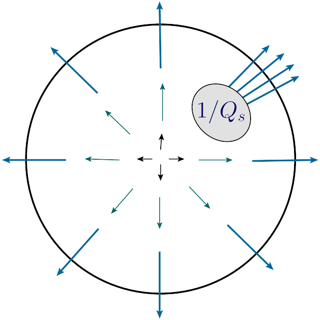
The Ridge in AA collisions

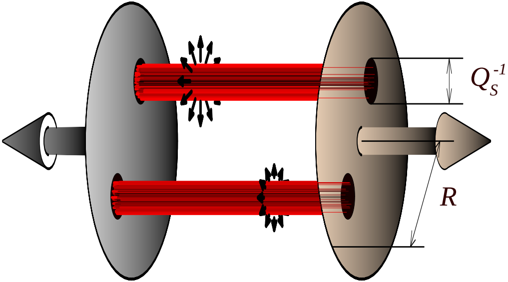
- "The ridge": di-hadron correlations long-ranged in $\Delta\eta$ & narrow in $\Delta\phi$
- Abundantly observed in AA collisions at both RHIC and the LHC
- ... and a surprize: for any $\Delta\eta$, the correlation is peaked at $\Delta\phi = 0$
  - particles moving along very different directions w.r.t. the beam axis preserve a common direction in the transverse plane





- The nuclear disk has a finite radius and it is denser towards its center
  - there is also a radial flow: particles move away from the center
  - the radial velocity v_r increases with the impact parameter b
- Particles from a same interaction region with size $\sim 1/Q_s$ feel the same radial push





  - they would be produced isotropically in the absence of flow

Radial flow

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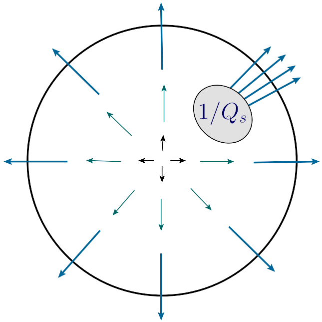
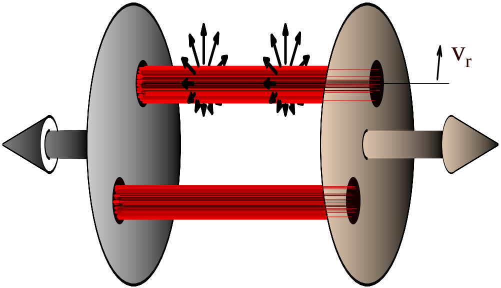
  - the radial velocity v_r increases with the impact parameter b
- Particles from a same interaction region with size $\sim 1/Q_s$ feel the same radial push

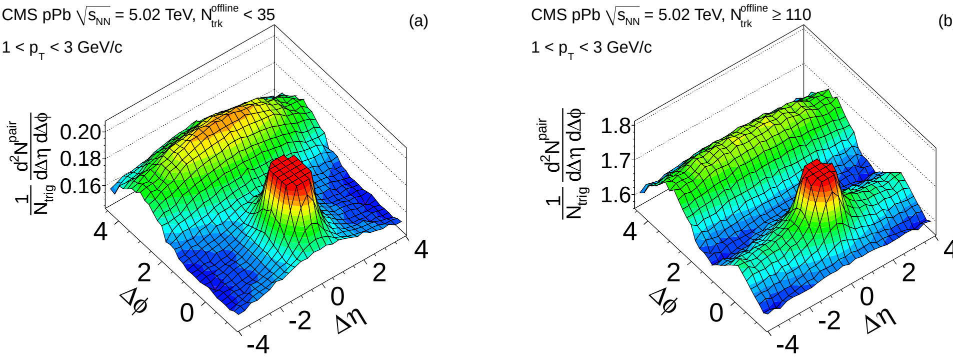




  - the radial flow introduces a bias leading to collimation in $\Delta\Phi$

The Ridge in pp and pA

- LHC : quite surprisingly, a ridge is also observed in p+p and p+A events with unusually high multiplicity



- Can flow develop in such small systems (~ 1 fm) ?
- If not ... this may reflect intrinsic momentum correlations at early times

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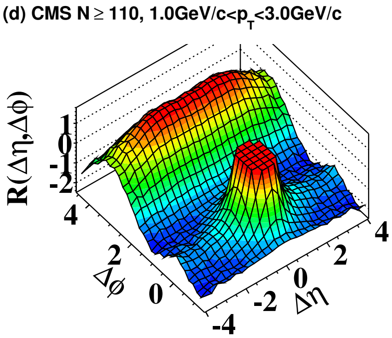

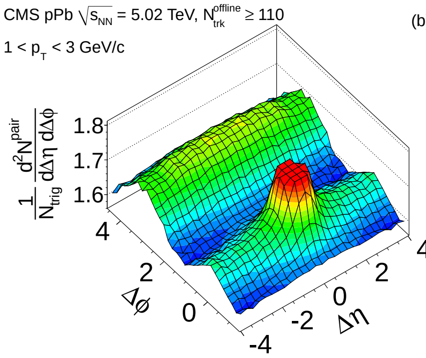

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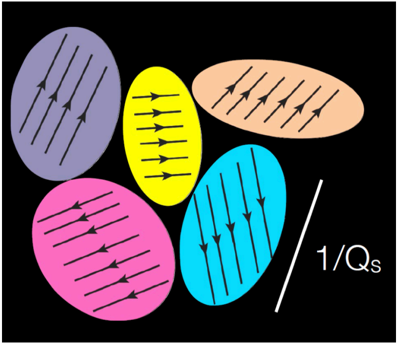
Angular correlations from saturation

- Dilute-dense scattering (say, pA): saturation domains in the dense target
  - color fields are correlated within domains with transverse area $\sim 1/Q_s^2$
  - in each event, different domains are randomly oriented w.r.t. each other
  - this domain structure fluctuates from event to event
- When 2 (or more) quarks from the projectile scatter off a same domain, they will receive a similar kick provided they are in the same color state



$$\delta k^i = qg\int dx^+ F^{i-}\,, \qquad |\delta \boldsymbol{k}| \sim Q_s$$

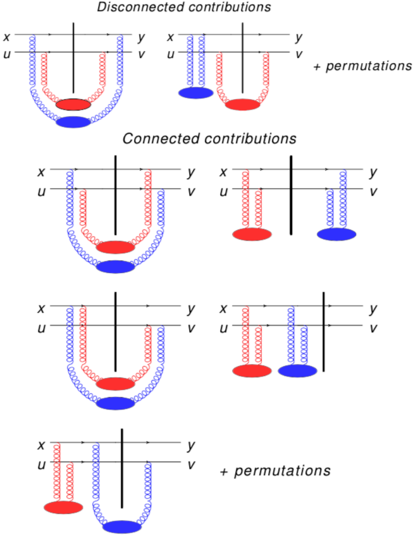
- $q = R,\, B,\, G$: quark color charge
- correlation suppressed as $1/N_c^2$
- ... and by the number of domains probed by the projectile

$$\mathcal{C}(\Delta\phi \simeq 0) \propto \frac{1}{N_c^2}\,\frac{1}{Q^2 S_{\rm proj}}$$

Glasma graphs for 2 quark production

(T. Lappi, B. Schenke, S. Schlichting and R. Venugopalan, arXiv: 1509.03499)

- Independent scattering ("disconnected") and correlations ("connected")



- one quark production

$$\frac{\mathrm{d}N}{\mathrm{d}^2\boldsymbol{k}} = \mathcal{S}(\boldsymbol{k}) \simeq \frac{1}{\pi Q_s^2}\, \mathrm{e}^{-k_\perp^2/Q_s^2}$$

- two quark production

$$\frac{\mathrm{d}N}{\mathrm{d}^2\boldsymbol{k}\mathrm{d}^2\boldsymbol{p}} = \mathcal{S}(\boldsymbol{k})\mathcal{S}(\boldsymbol{p})\Big\{1 + \mathcal{C}(\boldsymbol{k},\boldsymbol{p})\Big\}$$

$$\mathcal{C}(\boldsymbol{k},\boldsymbol{p}) \simeq \frac{\delta^{(2)}(\boldsymbol{k}-\boldsymbol{p}) + \delta^{(2)}(\boldsymbol{k}+\boldsymbol{p})}{S_{\mathrm{proj}} N_c^2}$$

- "ridge": $\boldsymbol{k} \simeq \boldsymbol{p}$, or $\Delta\phi \simeq 0$
- "elliptic flow": $\boldsymbol{k} \simeq -\boldsymbol{p}$, or $\Delta\phi \simeq \pi$