LOW FLUCTUATIONS
\[ \frac{dN}{d\phi} \propto 1 + \sum_{n=1}^{\infty} \frac{v_n}{2} \cos n(\phi - \psi_n) \]
\[ \Rightarrow \langle \frac{dN_{\text{pairs}}}{d\Delta \phi} \rangle \propto 1 + \sum_{n=1}^{\infty} \frac{v_n^2}{2} \cos n(\Delta \phi) \]

\[ \psi \propto e^{2i\Phi} - \frac{r^2 e^{2i\phi}}{r^2} \] (Holopainen, Niemi, Eskola, Phys. Rev. C83, 034901 (2011))
“Dense–dense scattering” : much more complicated!

Non–linear effects enter at all levels

- in both incoming wavefunctions: gluon saturation
- in the scattering process: multiple interactions
- in the partonic medium created by the early scattering: final–state interactions
“Dense–dense scattering” : much more complicated!

Non–linear effects enter at all levels

- 2 CGC weight functions: \( W_{Y_1}[\rho_1], W_{Y_2}[\rho_2] \)

- in the scattering process: multiple interactions

- in the partonic medium created by the early scattering: final–state interactions
“Dense–dense scattering”: much more complicated!

Non-linear effects enter at all levels:

- 2 CGC weight functions: $W_{Y_1}[\rho_1], W_{Y_2}[\rho_2]$
- Classical Yang–Mills equations with 2 sources: $\rho_1, \rho_2$
- In the partonic medium created by the early scattering: final–state interactions
“Dense–dense scattering”: much more complicated!

- Non-linear effects enter at all levels
  - 2 CGC weight functions: $W_{Y_1}[^{\rho_1}]$, $W_{Y_2}[^{\rho_2}]$
  - classical Yang–Mills equations with 2 sources: $\rho_1$, $\rho_2$
  - kinetic theory, hydrodynamics, quark-gluon plasma, ...
CGC factorization for AA collisions

- Numerically solve classical YM equations with 2 sources (2D lattice)

\[ D_\nu F^{\nu\mu}(x) = \delta^{\mu+}\rho_1(x) + \delta^{\mu-}\rho_2(x) \]

- Decompose the classical field \( A_\mu^a \) in Fourier modes

  - gluon spectrum for given configurations \( \rho_1 \) and \( \rho_2 \) ("event-by-event")

- Average over \( \rho_1 \) and \( \rho_2 \) using the CGC distributions of the nuclei

\[ \left\langle \frac{dN}{dY \, d^2p_\perp} \right\rangle = \int \left[ D\rho_1 D\rho_2 \right] W_{Y_{beam}^- \, Y_{beam}^+} \left[ \rho_1 \right] W_{Y_{beam}^+ \, Y_{beam}^-} \left[ \rho_2 \right] \frac{dN}{dY \, d^2p_\perp} \bigg|_{\text{class}} \]

  - JIMWLK evolution from \( Y_{beam} \) up to the rapidity \( Y \) of the produced gluon

![Diagram showing the CGC factorization and evolution](image-url)
What are the chromo-electric and magnetic fields created by an ultrarelativistic nucleus? \( \rho(x) \propto \delta(x^+) \) for a left mover

- non-Abelian generalization of the Liénard-Wiechert potentials
- Weiszäcker-Williams fields describing quasi-real photons/gluons

\[ E_a \perp B_a \perp z \]

\[ E_\perp \cdot B_\perp = 0, \quad |E_\perp| = |B_\perp| \sim \frac{1}{g} \]

- transverse polarizations
- chromo-electromagnetic waves
- Lorentz contraction: \( \propto \delta(x^+) \)

Fields vary over a distance \( \sim 1/Q_s \) \( \implies \) gluons typically have \( k_\perp \sim Q_s \)

Fields have strength \( \sim 1/g \) \( \implies \) gluons have occupation numbers \( \sim 1/\alpha_s \)
The scattering between two color sheets

- Prior to scattering: purely transverse fields, localized near the 2 light-cones
The scattering between two color sheets

- Prior to scattering: purely transverse fields, localized near the 2 light-cones.

- During collision, mutual color rotations induce color charges on the sheets:
  - Longitudinal chromo-electric and chromo-magnetic fields
  - Color strings (flux tubes) with typical transverse size $1/Q_s$
The ‘valence’ charges of the 2 nuclei rapidly separate from each other.

The color field between the 2 recessing nuclei: ‘glasma’ (‘glass’ + ‘plasma’)

After a time $\tau \sim 1/Q_s$, the transverse fields are regenerated.

By that time, the partonic system becomes dilute (field strengths become of $\mathcal{O}(1)$), due to longitudinal expansion.

Fourier mode decomposition $\rightarrow$ gluon production.
Gluons liberated by the collision at $t = 0$ have transverse momenta $k_\perp \sim Q_s$ and generic longitudinal momenta $|k_z| \sim x_1 P_1^+$ or $|k_z| \sim x_2 P_2^-$.

After a time $t \sim 1/Q_s$, they separate from each other along the $z$ axis.

- particles which at time $t$ are located at $z$ have a velocity $v_z = \frac{z}{t}$.

In any slice of $z$, the distribution in the transverse plane is roughly isotropic.

But the 3-dimensional distribution in momentum is highly anisotropic.
Rapidity correlations

- Consider pairs of particles in the final state with different rapidities
  - hadrons which propagate at different angles w.r.t. the collision axis

\[
\eta = \frac{1}{2} \ln \frac{p + p_z}{p - p_z} = \frac{1}{2} \ln \frac{1 + v_z}{1 - v_z} = -\ln \tan \frac{\theta}{2}
\]

- Construct the 2-hadron correlation in \( \eta \) and \( p_\perp \) (cf. talk by Jan Fiete):

\[
C(\Delta \phi, \Delta \eta) \equiv \frac{dN_{\text{pair}}}{d^2 p_\perp d\eta_1 d^2 p_\perp d\eta_2} - \frac{dN}{d^2 p_\perp d\eta_1} \frac{dN}{d^2 p_\perp d\eta_2}
\]

- interested in the distribution in \( \Delta \eta = \eta_1 - \eta_2 \) and in \( \Delta \phi = \phi_1 - \phi_2 \)

- Large \( \Delta \eta > 1 \) \( \implies \) particles with very different longitudinal velocities
  - they separate from each other after a time \( \tau \sim 1/Q_s \simeq 0.2 \text{ fm} \)
  - their correlations must have been built at smaller times \( \tau \lesssim 1/Q_s \)

- Long-range correlations in rapidity teach us about the early stages/glasma
Collective flow in AA collisions (cf. lecture by Jan Fiete)

- **Left:** p+Pb collisions: “trivial” correlations associated with jets
  - a peak around $\Delta \eta = 0$ and $\Delta \phi = 0$: particles from a same jet
  - a plateau in $\Delta \eta$ narrow in $\Delta \phi$ around $\Delta \phi = \pi$: recoiling jets

- **Right:** Pb+Pb collisions: long-range correlations in $\Delta \eta$ narrow in $\Delta \phi$
  - both on the “near side” ($\Delta \phi \approx 0$) and on the “away side” ($\Delta \phi \approx \pi$)

### Diagrams

1. **Left:** CMS pPb $\sqrt{s_{NN}} = 5.02$ TeV, $N_{\text{trk}}^{\text{offline}} < 35$
   - $1 < p_T < 3$ GeV/c

2. **Right:** CMS PbPb $\sqrt{s_{NN}} = 2.76$ TeV, $220 \leq N_{\text{trk}}^{\text{offline}} < 260$
   - $1 < p_T^{\text{trig}} < 3$ GeV/c
   - $1 < p_T^{\text{assoc}} < 3$ GeV/c

---

Elliptic flow

What is the origin of the double peak structure ($\Delta \phi = 0$ and $\pi$)?

Maximum at $\Delta \phi = 0$ or $\pi$; minimum at $\Delta \phi = \pi/2$: elliptic flow

$R \equiv \frac{\langle N_1 N_2 \rangle - \langle N_1 \rangle \langle N_2 \rangle}{\langle N_1 \rangle \langle N_2 \rangle} \propto v_2^2 \cos (2\Delta \phi)$
The geometry of a HIC

Number of participants ($N_{\text{part}}$): number of incoming nucleons (participants) in the overlap region

"peripheral" collision ($b \sim b_{\text{max}}$)
"central" collision ($b \sim 0$)

Reaction plane
Elliptic flow $v_2$

\[ \frac{dN}{d\phi} \propto 1 + 2v_2 \cos 2\phi \]

$v_2$: the ‘coefficient of the elliptic flow’

- Non-central $AA$ collision: impact parameter $b_\perp > 0$
- The interaction region is (roughly) elliptic
- Pressure gradient is larger along the smaller axis ($x$)
- Fluid velocity is proportional to the pressure gradient
- Particle emerge predominantly parallel to the fluid velocity

$\implies$ the particle distribution is not axially symmetric!
Elliptic flow $v_2$

$$\frac{dN}{d\phi} \propto 1 + 2v_2 \cos 2\phi$$

$v_2$ : the ‘coefficient of the elliptic flow’

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- Particle emerge predominantly parallel to the fluid velocity
- Initial anisotropy in coordinate space yields final anisotropy in momentum
Granularity and fluctuations

- Nucleons are randomly distributed inside a nucleus.
- In some events, the shape of the interaction (overlap) region can be quite different from an ellipse
- Then one speaks about triangular flow (or higher harmonics)

\[ \frac{dN}{d\phi} \propto 1 + 2v_2 \cos 2(\phi - \Psi_2) + 2v_3 \cos 3(\phi - \Psi_3) + \ldots \]

- The small disks need not be nucleons: they can also be saturation domains
- \( \Delta x_\perp \sim 1/Q_s \sim 0.2 \text{ fm} \ll R_N \sim 1 \text{ fm} \)
Centrality dependence for $v_2$ and $v_3$

**NEW FLOW OBSERVABLES**: $v_3$, $v_4$, $v_5$, ...

**ALICE, arXiv:1105.3865**

\[
\varepsilon_2 = \frac{\langle y^2 - x^2 \rangle}{\langle y^2 + x^2 \rangle}
\]

- **Ultra-central collisions 0 ÷ 5%**: the more central, the smaller $v_2$
  - central collisions: the interaction region is axially symmetric
  - one expects $v_2$ to be proportional to the eccentricity
  - CGC calculations of the initial $\varepsilon_2$ compare better than just a random distribution of “wounded” nucleons (Glauber)
- $v_3$ is much less sensitive to the centrality: controlled by fluctuations

CSS:

Glauber may not work either

MATT LUZUM (IPHT)
LOW FLUCTUATIONS
QUARK MATTER
2011 9 / 13
Momentum dependence for $v_2$

$v_2$ first rises up to $3 \div 4$ GeV, then it decreases again

- the flow is collective motion which contributes to $p_\perp$
- sufficiently fast particles cannot be driven by the flow

- The high-$p_\perp$ tail: natural when fluctuations occur on smaller scales: $1/Q_s \ll R_N$
$p_\perp$ dependence for $v_n$, $n = 2 - 6$ (ATLAS)

Similar $p_\perp$ dependence for all $n$: rise up to 3-4 GeV, then fall
To describe flow, it is natural to use hydrodynamics:

- transport of conserved quantities (energy, momentum, electric and baryonic charges ...) via collective motion with fluid velocity $v$

This is perhaps best introduced by comparing with thermodynamics:

- Thermodynamics is about a system in global thermal equilibrium
  - pressure ($P$), temperature ($T$), chemical potential ($\mu$) are independent of time and uniform throughout the whole volume $V$

- Hydrodynamics is about local thermal equilibrium
  - $P$, $T$, $\mu$ and $v$ can vary with space and time ...
  - ... but they vary so slowly that one can still assume thermal equilibrium to hold locally, in the neighbourhood of any point
  - equations of motion for $P(x)$, $T(x)$, $\mu(x)$, $v(x)$

This holds when the typical scale for space–time variations ('system size $R$') is much larger than the mean free path $\ell$ between two successive collisions.
To describe flow, it is natural to use hydrodynamics

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Gradient expansion in powers of $K \equiv \ell/L \ll 1$ (Knudsen number)

In heavy ion collisions, $L \sim R_A$: transverse size of the nucleus
Ideal Hydrodynamics

- Hydro equations of motion: the conservation laws
  \[ \partial_\mu T^{\mu\nu} = 0 \quad \partial_\mu J_B^\mu = 0 \]

- \( T^{\mu\nu} \) (energy-momentum tensor) and \( J_B^\mu \) (baryonic current):
  - fluid 4-velocity: \( u^\mu(x) = \gamma(1, v) \), \( \gamma = 1/\sqrt{1-v^2} \)
  - energy density \( \epsilon(x) \), pressure \( P(x) \), baryon density \( n_B(x) \)
  - additional parameters (‘viscosities’) for a non-ideal fluid

- Ideal (“inviscid”) fluid \( \equiv \) local thermal equilibrium \( \rightarrow \) isotropy

- \( T^{\mu\nu}_{(0)} = \text{diag}(\epsilon, P, P, P) \) in the local rest frame at \( x: u^\mu(x) = (1, 0) \)

- After a boost to the laboratory frame, this becomes:
  \[ T^{\mu\nu}_{(0)} = (\epsilon + P)u^\mu u^\nu - Pg^{\mu\nu}, \quad J_B^\mu = nu_B^\mu \]

- \( T^{\mu\nu} \): 4 equations, 5 unknown quantities: \( \epsilon(x), P(x), v^i(x) \) (\( i = 1, 2, 3 \))

- Use the equation of state \( \epsilon = \epsilon(P) \) to close the system
Hydro equations of motion: the conservation laws

\[ \partial_\mu T^{\mu\nu} = 0 \]

\[ \partial_\mu J^\mu_B = 0 \]

\( T^{\mu\nu} \) (energy-momentum tensor) and \( J^\mu_B \) (baryonic current):

- fluid 4-velocity: \( u^\mu(x) = \gamma(1, v) \), \( \gamma = \frac{1}{\sqrt{1-v^2}} \)
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\( T^{\mu\nu} \): 4 equations, 5 unknown quantities: \( \epsilon(x), P(x), v^i(x) \) \( i = 1, 2, 3 \)

E.g.: ideal gas of massless particles: \( \epsilon = 3P \), hence \( T^\mu_{\mu} = 0 \)
Initial conditions & the thermalization puzzle

- One needs the initial conditions $\epsilon_0(\vec{x})$, $v_0^i(\vec{x})$ at some proper time $\tau_0$
- The isotropy seems inconsistent with the geometry of a heavy ion collision
  - strong dissymmetry between longitudinal and transverse directions

\[ T_{YM}^{\mu\nu} = \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - F^{\mu}_{\alpha} F^{\nu\alpha} \]

- $\tau = 0^+$: purely longitudinal fields

\[ T_{YM}^{\mu\nu} = \frac{1}{2} (E_z^2 + B_z^2) \times \text{diag}(1, 1, 1, -1) \]
- negative “longitudinal pressure”

- $\tau \sim 1/Q_s$: $T_{YM}^{\mu\nu} = \text{diag}(\epsilon, \epsilon/2, \epsilon/2, 0)$: zero “longitudinal pressure”
- “Rapid thermalization at some time $\tau_0 \lesssim 1 \text{ fm}$” (?)
Viscous hydrodynamics

- Deviations from local thermal equilibrium are captured by viscous hydro.
- Ideal hydro: zeroth order in the gradient expansion $\Rightarrow$ no dissipation.
- Dissipative effects are described by higher orders: $T^{\mu\nu} = T_{(0)}^{\mu\nu} + \Pi^{\mu\nu}$
- The first-order terms: $\partial^\mu u^\nu$ or $g^{\mu\nu}(\partial \cdot u)$ are weighted by the viscosities.
  - shear viscosity $\eta$, bulk viscosity $\zeta$: “transport coefficients”
  - measure the fluid ability to transfer energy and momentum along directions orthogonal to that of the flow.
  - heuristically: friction between the different layers of the fluid.
- Non-relativistic limit $v \ll 1 \Rightarrow$ Navier-Stokes equation ($\rho = \text{mass density}$)

$$\rho \left( \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} \right) \vec{v} = -\vec{\nabla}P + \eta \nabla^2 \vec{v} + \zeta \vec{\nabla} (\vec{\nabla} \cdot \vec{v})$$

- First-order relativistic hydro is inconsistent with causality, but this can be fixed by adding second-order terms (“Müller-Israel-Stewart theory”).
Shear viscosity

- 2 layers of a same fluid, one flowing the other one at rest, separated by a wall
- Remove the wall: the two layers would not get mixed for ideal flow ($\eta = 0$)
- $\eta$: a measure of a fluid ability to transfer $p_x$ in the $y$ direction

$$\frac{1}{A} \frac{dp_x}{dt} = -\eta \frac{du_x}{dy}$$

$$\eta \sim \ell \rho v_{th} \sim \ell \epsilon / \rho v_{th}$$

- $v_{th}$ average molecular velocity ($\gg$ fluid velocity $v$)

- Proportional to the mean free path: $\ell \propto 1/(n\sigma) \sim 1/\alpha_s^2$
- larger for weakly coupled systems! (Maxwell, 1860)
Initial conditions: from smoothness to granularity

- Left: mean field description — a smooth distribution for the nuclear matter

**Optical Glauber**

- Thickness functions: $T_{A,B}(s) = \int_{-\infty}^{\infty} dz \rho_{A,B}(z,s)$
- Overlap function: $T_{AB}(b) = \int d^2s T_A(s) T_B(s-b)$
- Number of collisions: $N_{\text{coll}}(b) = A B T_{AB}(b) \sigma_{\text{NN}}^{\text{inel}}$
- Number of participants: $N_{\text{part}}(b) = A \int d^2s T_A(s) (1 - [1 - T_B(s-b) \sigma_{\text{NN}}^{\text{inel}}] A)$ + $B \int d^2s T_B(s-b) (1 - [1 - T_A(s) \sigma_{\text{NN}}^{\text{inel}}] B)$

- Smooth distribution $\Rightarrow$ No fluctuations.
- No $\epsilon_{\text{odd}}$ nor $v_{\text{odd}}$

- Right: random distribution of the nucleons inside each nucleus
  $\Rightarrow$ fluctuations (Monte-Carlo Glauber)
Initial conditions: from nucleons to Glasma

- **Left:** random superposition of nucleons within the nuclear disks
  
  ![Graphical representation of nucleon distribution](image)

  - Monte-Carlo Glauber (nucleons)
  - size of fluctuations: $R_p \sim 1$ fm

- **Right:** Glasma simulation (classical Yang–Mills equations with randomly distributed color charges) *(see review by Gelis and Schenke, 2016)*

  ![Graphical representation of Glasma simulation](image)

  - Glasma (color flux tubes)
  - size of fluctuations: $1/Q_s \sim 0.2$ fm
Early simulations used ideal hydro + Monte-Carlo Glauber (nucleons)

- required early equilibration $\tau_0 \lesssim 1 \text{ fm}/c$: very puzzling

Glasma generates a larger initial eccentricity $\varepsilon_2$

- this leaves the place for a larger viscosity, i.e. for deviations from local thermal equilibrium

The CGC fits prefer a larger value for the viscosity/entropy density ratio
Hydro simulations for $v_2$ \cite{Gelis_2016}

- Early simulations used ideal hydro + Monte-Carlo Glauber (nucleons)
  - required early equilibration $\tau_0 \lesssim 1$ fm/c: very puzzling

- Glasma generates a larger initial eccentricity $\varepsilon_2$

- Viscous hydro + Glasma yields very good fits to all $v_n$'s with $\eta/s = 0.2$

- This value $\eta/s$ seems still too small to be consistent with weak coupling
Viscosity over entropy density ratio

- $\eta \sim \ell \times \epsilon$ ($\ell$: mean free path; $\epsilon$: energy density)

- Thermodynamics: $\epsilon + P = Ts \implies \epsilon/s \sim T \sim 1/\lambda_{th}$

- Uncertainty principle $\implies$ ideal fluids ($\eta = 0$) cannot exist in nature

$$\frac{\eta}{s} \sim \frac{\ell}{\lambda_{th}} \sim \frac{\text{mean free path}}{\text{de Broglie wavelength}} \gtrsim \hbar$$

- Weakly coupled QGP (Arnold, Moore, Yaffe, 2003):

$$\frac{\eta}{s} \sim \frac{\hbar}{\alpha_s^2 \ln(1/\alpha_s)} \gg \hbar$$

- Conjectured limit at strong coupling (Kovtun, Son, Starinets, 05)

$$\frac{\eta}{s} \rightarrow \frac{\hbar}{4\pi} \quad \text{when} \quad \lambda \equiv g^2 N_c \rightarrow \infty \quad (\text{AdS/CFT})$$

- The value preferred by the fits is at most a few times $1/4\pi \approx 0.08$
RHIC serves up the perfect liquid!

RHIC Scientists Serve Up "Perfect" Liquid

New state of matter more remarkable than predicted -- raising many new questions

Monday, April 18, 2005

TAMPA, FL -- The four detector groups conducting research at the Relativistic Heavy Ion Collider (RHIC) -- a giant atom "smasher" located at the U.S. Department of Energy's Brookhaven National Laboratory -- say they've created a new state of hot, dense matter out of the quarks and gluons that are the basic particles of atomic nuclei, but it is a state quite different and even more remarkable than had been predicted. In peer-reviewed papers summarizing the first three years of RHIC findings, the scientists say that instead of behaving like a gas of free quarks and gluons, as was expected, the matter created in RHIC's heavy ion collisions appears to be more like a liquid.

- ‘Strongly–coupled quark–gluon plasma’ or ‘perfect fluid’

- Applications of the AdS/CFT correspondence to heavy ion collisions

- Two main arguments in favor of strong coupling:
  - small $\eta/s$ ratio from the hydro fits to the flow coefficients
  - small thermalization time needed for consistency of hydro
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- ‘Strongly–coupled quark–gluon plasma’ or ‘perfect fluid’
- Applications of the AdS/CFT correspondence to heavy ion collisions
- Two main arguments in favor of strong coupling:
  - small $\eta/s$ ratio from the hydro fits to the flow coefficients
  - small thermalization time needed for consistency of hydro
- These 2 conditions can also be satisfied, or circumvented, at weak coupling
A weakly coupled QGP in thermal equilibrium with temperature $T$

**Thermal (quasi)particles:** quarks and gluons with energy and momenta of order $T$, whose properties are slightly dressed by the interactions

Number density $n \sim T^3$, energy density $\epsilon \sim T^4$

Mean free path: typical distance/time between two successive collisions

$$\ell \sim \frac{1}{\Gamma}, \quad \Gamma = \text{collision rate} = n\sigma v_{rel} \sim T^3\sigma$$

$\sigma$: cross-section for binary collisions

Weak coupling $\implies$ the dominant mechanism is $2 \rightarrow 2$ elastic scattering
Coulomb scattering (one gluon exchange), deflection angle \( \theta \approx \frac{q_\perp}{p} \)

\[
\frac{d\sigma}{dq_{\perp}^2} \sim \frac{\alpha_s^2}{q_{\perp}^4}
\]

the total cross-section

\[
\sigma = \int dq_{\perp}^2 \frac{\alpha_s^2}{q_{\perp}^4}
\]

... is infrared \((q_{\perp} \to 0)\) divergent
Mean free path at weak coupling (2)

- Coulomb scattering (one gluon exchange), deflection angle $\theta \sim q_\perp/p$

$$d\sigma /dq_\perp^2 \sim \alpha_s^2 /q_\perp^4$$

- the total cross-section

$$\sigma = \int_{\alpha_s T^2} dq_\perp^2 \frac{\alpha_s^2}{q_\perp^4} \sim \frac{\alpha_s}{T^2}$$

- In the plasma, color interactions are screened over a distance $\lambda_D \sim 1/gT$, that is, at the momentum scale $q_\perp^2 \sim \alpha_s T^2$ (Debye screening)

$$\ell_c \sim 1/(\alpha_s T) = \text{m.f.p. for small-angle scattering: } \theta \sim gT/T \sim g$$

- also for color relaxation: even soft collisions can change the color state

- Not the relevant m.f.p. for processes which require momentum transfer, like viscosity: large-angle scattering
Mean free path at weak coupling (3)

- The transport cross-section: weighted by $1 - \cos \theta \simeq \theta^2/2$ with $\theta \simeq q_\perp/p$

$$
\sigma_{tr} = \int dq_\perp \frac{q_\perp^2}{p^2} \frac{\alpha_s^2}{q_\perp^4}
$$

- Logarithmically sensitive to all “semi-hard” scales $gT \ll q_\perp \ll T$

$$
\ell \sim \frac{1}{\sigma_{tr} T^3} \sim \frac{1}{\alpha_s^2 T \ln(1/\alpha_s)} = \text{m.f.p. for large-angle scattering}
$$

- $\ell \gg \ell_c$: changing momentum is more difficult than changing the color

- Shear viscosity at weak coupling: $\eta \sim \ell \epsilon \sim \frac{T^3}{\alpha_s^2 \ln(1/\alpha_s)}$
Left: a compilation of the leading-order result (Arnold, Moore, Yaffe, 2003) by D. Teaney, arXiv:0905.2433

- fixed coupling & running coupling with 1-loop and 1-loop beta function

Right: the recent NLO calculation vs to the LO result (2-loop running) Ghiglieri, Moore and Teaney, arXiv:1802:09535 (based on kinetic theory)
**η/s in pQCD: from LO to NLO**

- **Left:** a compilation of the leading-order result *(Arnold, Moore, Yaffe, 2003)* by D. Teaney, arXiv:0905.2433
  - fixed coupling & running coupling with 1-loop and 1-loop beta function

- **Right:** the recent NLO calculation vs to the LO result (2-loop running)
  - at $T \sim 300$ MeV, the result is reduced by a factor of 4: $\eta/s \sim 0.2$
Kinetic theory in a nut shell

- For weakly coupled systems, kinetic theory can be used to describe the non-equilibrium evolution at intermediate times $1/Q_s \lesssim \tau \lesssim \tau_0$

- Follow the non-equilibrium evolution of the phase-space occupation numbers

$$f_s(t, x, p) \equiv \frac{(2\pi)^3}{\nu_s} \frac{dN_s}{d^3xd^3p}$$

- $s = g, q, \bar{q}...; \nu_g = 2(N_c^2 - 1), \nu_q = \nu_{\bar{q}} = 2N_c$

- Meaningful when $f_s \ll 1/\alpha$ (which includes $f \gg 1$ at weak coupling)

- It smoothly matches onto classical Yang-Mills dynamics when $f \gg 1$ and onto hydro when off-equilibrium deviations are small: $\delta f \ll f_{eq}$

- All the observables can be computed from $f_s(t, x, p)$

$$T^{\mu\nu}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{p^\mu p^\nu}{E_p} \sum_s \nu_s f_s(t, x, p)$$
Boltzmann equation

- Two types of interactions: external force and collisions among the particles

\[
\left( \frac{\partial}{\partial t} + v_p \cdot \frac{\partial}{\partial x} + F_{\text{ext}} \cdot \frac{\partial}{\partial p} \right) f(t, x, p) = -C[f]
\]

- \(C[f]\): “collision term”: non-linear in \(f\), non-local in \(p\), but local in \(x\)

- Variation time scales much larger than the duration of the collision process
  - e.g. \(\Delta t \gg 1/(gT)\) for \(2 \rightarrow 2\) collisions near equilibrium

- \(C[f_{\text{eq}}] = 0\): collision term vanishes in thermal equilibrium
  - \(f_s(t, x, p) \rightarrow n_s(p)\)

\[
n_g(p) = \frac{1}{e^{\beta p} - 1}, \quad n_q, \bar{q}(p) = \frac{1}{e^{\beta(E_p + \mu)} + 1}
\]

- Bose-Einstein for gluons, Fermi-Dirac for quarks

- Collisions can be either elastic (e.g. \(2 \rightarrow 2\)) or inelastic (e.g. \(2 \rightarrow 3\))
Elastic collisions

- **2 → 2 collisions:** generally argued to be the LO piece at weak coupling

  \[
  p \rightarrow \quad k \\
  q \quad \quad -q \\
  p' \rightarrow \quad k' \\
  
  \]

- **‘Gain’ and ‘loss’ terms:** the gluon \( p \) is the one that is measured

  \[
  C_{\text{el}}[f] = \int_{p',k,k'} \frac{|M|^2}{(2p)(2p')(2k)(2k')} \Phi[f] 
  \]

  \[
  \Phi[f] = f_p f_{p'} [1 + f_k][1 + f_{k'}] - f_k f_{k'} [1 + f_p][1 + f_{p'}] 
  \]

- **5 conserved quantities:** particle number and the 4–momentum

- It vanishes at (local) thermal equilibrium: \( C_{\text{el}}[n] = 0 \) (“detailed balance”)

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Elastic collisions

- \(2 \rightarrow 2\) collisions: generally argued to be the LO piece at weak coupling

\[\begin{array}{c}
\begin{array}{ccc}
\vec{p} & \vec{k} & \vec{p}' \\
\downarrow & \downarrow & \downarrow \\
q & \vec{k}' & -q \\
\end{array}
\end{array}\]

- ‘Gain’ and ‘loss’ terms: the gluon \(\vec{p}\) is the one that is measured

\[\mathcal{C}_{\text{el}}[f] = \int_{p',k,k'} \frac{|\mathcal{M}|^2}{(2p)(2p')(2k)(2k')} \Phi[f]\]

\[\Phi[f] = f_p f_{p'} [1 + f_k][1 + f_{k'}] - f_k f_{k'} [1 + f_p][1 + f_{p'}]\]

- It also vanishes in the limit of zero momentum transfer \(q = |q| \rightarrow 0\)

- transport cross-section: weighted by a factor \(q^2\) at small \(q\)
The thermalization puzzle

- An off-equilibrium perturbation typically dies away over a time $\tau_{\text{rel}} \sim \ell/v_{\text{th}}$

- At weak coupling, $\tau_{\text{rel}} \sim \ell$ is parametrically large (in units of $\lambda_{\text{th}} = 1/T$)

$$\frac{\tau_{\text{rel}}}{\lambda_{\text{th}}} \sim \frac{\eta}{s} \sim \frac{1}{\alpha_s^2 \ln(1/\alpha_s)} \gg 1$$

- The folklore says: “Fast thermalization is not possible at weak coupling”

- Not quite true: recall the NLO calculation of $\eta/s$ (Ghiglieri et al, arXiv:1802:09535)
  - at $T \sim 300$ MeV, the result is reduced by a factor of 4: $\eta/s \sim 0.2$

- The actual problem is the rapid longitudinal expansion
  - the partonic system becomes more and more dilute
  - collisions occur less often and have little chances to equilibrate

- A solution to that is provided by radiation (“inelastic collisions”)

New Trends in High-Energy Physics
Glasma, Flow & Thermalization

Edmond Iancu
Anisotropy from the longitudinal expansion

At time $t \sim 1/Q_s$, gluons are freed from the nuclei with $p_\perp \sim Q_s$ ("hard") and occupation numbers $f_0 \sim 1/\bar{\alpha}$.

At later times $t \gg 1/Q_s$, all the gluons located at $z$ have $v_z = z/t$.

The distribution in velocity (or momentum) is highly anisotropic.

Focus on midrapidity for definiteness: $\eta \sim 0$ (say, $z \sim 1/Q_s$).

Physics is boost invariant: replace $t \rightarrow \tau \equiv \sqrt{t^2 - z^2}$ for any other $\eta$.

$$v_z \sim \frac{p_z}{p_\perp} \sim \frac{1}{Q_s \tau} \quad \Rightarrow \quad p_z \sim \frac{1}{\tau} \ll p_\perp \sim Q_s \quad \text{when} \quad Q_s \tau \gg 1$$
The “Bottom-up” scenario \textit{(Baier, Mueller, Schiff, Son ’01)}

- This would be the distribution in the absence of collisions
- Elastic $2 \rightarrow 2$ scattering competes with the longitudinal expansion
  - $p_z$ decreases less fast with $\tau$: $p_z \sim \frac{Q_s}{(Q_s \tau)^{1/3}}$

The color scale: the occupation number in the range $0.1 < f < 5$

- numerical calculation by Kurkela et al, arXiv:1805.00961
- gluon distribution in the $(p_x, p_z)$ plane as a function of the scaled time (with $\eta/s = 0.62$)

\[
\frac{\tau}{\tau_{\text{rel}}(\tau) \equiv \frac{\tau T(\tau)}{\eta/s}}
\]

- Anisotropy less pronounced than for free streaming, but steadily increasing
The “Bottom-up” scenario \((Baier, Mueller, Schiff, Son '01)\)

- The “hard” \((p_\perp \sim Q_s)\) gluons can also suffer “inelastic collisions”
  - collisions which trigger emissions of softer gluons
- The soft gluons are emitted isotropically
  - they escape the problem of the longitudinal expansion
- They can thermalize, over a time \(\sim \tau_{\text{rel}}\), via elastic \(2 \rightarrow 2\) scattering
The “Bottom-up” scenario (Baier, Mueller, Schiff, Son ’01)

- The “hard” \((p_\perp \sim Q_s)\) gluons can also suffer “inelastic collisions”
  - collisions which trigger emissions of softer gluons
- The hard particles scatter off the soft thermal both
- They efficiently lose energy via medium-induced radiation
- The distribution approaches local thermal equilibrium

\[
\frac{\tau_{\text{idi}}}{4\pi\eta/s} \approx 0.1 \quad p_x \\
\frac{\tau_{\text{idi}}}{4\pi\eta/s} \approx 0.5 \quad p_x \\
\frac{\tau_{\text{idi}}}{4\pi\eta/s} \approx 1.0 \quad p_x
\]
Medium-induced branchings

- The prototype: \( 2 \leftrightarrow 3 \) inelastic scattering (single scattering)

- Both splittings \((2 \to 3)\) and recombinations \((3 \to 2)\)
- The splittings (recombinations) can contribute both to “gain” and to “loss”
- Actually, multiple scattering is important (“LPM effect”)
Hydrodynamisation in the “Bottom-up” scenario


- Numerical solutions to the Boltzmann equation with $C[f] = C_{el}[f] + C_{inel}[f]$

- Highly anisotropic initial conditions: $P_L \ll P_T$

\[\bar{T}_{\mu\nu}/\bar{T}_{\mu\nu} \text{id.} = \frac{\lambda}{\tau_{T\text{id.}}/(4\pi\eta/s)}\]

- different values of the coupling ($\eta/s$)

\[\lambda \equiv g^2 N_c = 4\pi \alpha_s N_c\]

- viscous hydro (2nd order in gradients)

- hydro: universal curve of the rescaled time

\[\frac{\tau}{\tau_{\text{rel}}(\tau)} \equiv \frac{\tau T(\tau)}{\eta/s}\]

- Kinetic theory $\Rightarrow$ universal curve quasi-independent of the coupling (or $\eta/s$)

- Perfect matching with viscous hydro for $\tau \gtrsim \tau_{\text{rel}}$

- ... albeit the anisotropy is still pronounced: $(P_T - P_L)/P_T \sim 30\%$
Hydrodynamisation in the “Bottom-up” scenario


- Numerical solutions to the Boltzmann equation with $C[f] = C_{el}[f] + C_{inel}[f]$

- Highly anisotropic initial conditions: $P_L \ll P_T$

\[ \lambda \equiv g^2 N_c = 4\pi \alpha_s N_c \]

- different values of the coupling ($\eta/s$)

- viscous hydro (2nd order in gradients)

- hydro: universal curve of the rescaled time

\[ \frac{\tau}{\tau_{rel}(\tau)} \equiv \frac{\tau T(\tau)}{\eta/s} \]

- The fact that the data for the flow coefficients $v_n$ can be well described by viscous hydro does not mean that the QGP has reached local equilibrium!
Hydrodynamisation in the “Bottom-up” scenario


- Numerical solutions to the Boltzmann equation with \( C[f] = C_{\text{el}}[f] + C_{\text{inel}}[f] \)
- Highly anisotropic initial conditions: \( P_L \ll P_T \)

\[ \tau_{\text{id}}/(4\pi \eta/s) \]

\[ \frac{\tau}{\tau_{\text{rel}}(\tau)} \equiv \frac{\tau T(\tau)}{\eta/s} \]

- different values of the coupling \((\eta/s)\)
  \[ \lambda \equiv g^2 N_c = 4\pi \alpha_s N_c \]
- viscous hydro (2nd order in gradients)
- hydro: universal curve of the rescaled time

In proper units, the results are independent of the coupling

In fact, the same conclusion (hydrodynamisation w/o thermalization) has been reached at infinite coupling, within AdS/CFT (Chesler and Yaffe, 2008)
A more realistic simulation of a heavy ion collision

- initial conditions at $\tau = 0_-$: McLerran-Venugopalan model
- numerical solutions to classical Yang-Mills equations (glasma)
- negative longitudinal pressure at times $0_+ < \tau < 1/Q_s = 0.1$ fm
- matching onto kinetic theory at $\tau = 0.2$ fm
- matching onto viscous hydro at $\tau = 2$ fm
THANK YOU FOR YOUR ATTENTION!

... and have a good trip back home!