QCD and hadron structure
Part 5: TMDs

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Measured transverse momentum

- consider
  - Drell-Yan with measured small $q_T$ of $\gamma^*$
  - SIDIS with measured small $q_T$ of hadron
  - $e^+e^- \rightarrow h_1h_2 + X$ with $h_1, h_2$ approx. opposite momenta and small relative $q_T$

- $k_T \sim m$ from collinear graphs matters in final state
  - can still neglect parton $k_T$ in hard scattering
  - but do not $\int d^2k_T$ in parton densities and fragm. fcts.

- $k_T$ dependent/unintegrated PDFs
  - also called TMDs (transverse-momentum distributions)

- theoretical framework: TMD factorization
  - also called $k_T$ factorization
    - different from (but related to) $k_T$ factorization at small $x$
\( k_T \) dependent parton densities

\( k_T \) integrated:

\[
f_1(x) = \int d\z^- \frac{e^{i\z^-p^+x}}{4\pi} \langle p, s | \bar{q}(0) \gamma^+ W(0, \infty) W(\infty, z^-) q(z^-) | p, s \rangle \bigg|_{z^+=0, z_T=0}
\]

\( k_T \) dependent:

\[
\int d\z^- \frac{d^2 z_T}{(2\pi)^2} e^{i\z^-p^+x} e^{-i\vec{k}_T \cdot \vec{z}_T} \langle p, s | \bar{q}(0) \gamma^+ W(0, \infty) W(\infty, z^-, z_T) q(z^-, z_T) | p, s \rangle \bigg|_{z^+=0}
\]

▶ fields at different transv. positions
implications on Wilson lines \( \rightarrow \) later
\( k_T \) dependent parton densities

\( k_T \) integrated:

\[
f_1(x) = \int \frac{dz^-}{4\pi} e^{iz^-p^+x} \langle p, s | \bar{q}(0)\gamma^+ W(0, \infty) W(\infty, z^-) q(z^-) | p, s \rangle \bigg|_{z^+ = 0, z_T = 0}
\]

\( k_T \) dependent:

\[
\int \frac{dz^-}{4\pi} \frac{d^2 z_T}{(2\pi)^2} e^{iz^-p^+x} e^{-i k_T z_T} \langle p, s | \bar{q}(0)\gamma^+ W(0, \infty) W(\infty, z^-, z_T) q(z^-, z_T) | p, s \rangle \bigg|_{z^+ = 0}
\]

\[
= f_1(x, k_T^2) - \frac{\epsilon^{ij} k_T S^j_T}{m} f_{1T}^\perp(x, k_T^2)
\]

\( \epsilon^{12} = -\epsilon^{21} = 1 \)

\( \epsilon^{11} = \epsilon^{22} = 0 \)

- fields at different transv. positions
  implications on Wilson lines \( \rightarrow \) later

- correlations between spins and transv. momentum
  e.g. Sivers function \( f_{1T}^\perp \)

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QCD and hadron structure
A zoo of distributions

- **Collinear twist 2 densities:**
  - $f_1$ unpol. quark in unpol. proton
  - $g_1$ correlate $s_L$ of quark with $S_L$ of proton
  - $h_1$ correlate $s_T$ of quark with $S_T$ of proton

- **$k_T$ dependent twist 2 densities:**
  - $f_1, g_1, h_1$ as above
  - $f_{1T}^\perp$ correlate $k_T$ of quark with $S_T$ of proton (Sivers)
  - $h_{1T}^\perp$ correlate $k_T$ and $s_T$ of quark (Boer-Mulders)
  - $g_{1T}, h_{1T}^\perp, h_{1L}^\perp$ three more densities

- **Analogous for fragmentation functions:**
  - $f_1 \leftrightarrow D_1$ unpolarized
  - $h_1^\perp \leftrightarrow H_1^\perp$ Collins fragm. fct.
TMD factorization (SIDIS as example)

- take $Q$ large and $q_T$ small ($\sim m$ for power counting purposes)

- transverse-momentum dep't distribution and fragmentation fcts.

- only virtual corrections to hard subgraph
  no radiation of high-$p_T$ partons allowed
TMD factorization (SIDIS as example)

- take $Q$ large and $q_T$ small ($\sim m$ for power counting purposes)

- soft gluon exchange does not cancel in sum over hadronic final state at leading-power accuracy gives soft factor in factorization formula

$S = \text{universal non-perturbative fct}$

$\rightarrow 1$ when integrate over $k_T$

cancellation between real
TMD factorization (SIDIS as example)

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$S = \text{universal non-perturbative fct}$

$\rightarrow 1$ when integrate over $k_T$

cancellation between real and virtual graphs
SIDIS at low $q_T$

- factorization formula

$$\frac{d\sigma_{\gamma^* p}}{dz \, dq_T^2} = (\text{kin. fact.}) \times |H(\mu)|^2 \int d^2 p_T \, d^2 k_T \, d^2 l_T \, \delta^{(2)}(\vec{p}_T - \vec{k}_T + \vec{l}_T + \vec{q}_T) \times \sum_{i=q,\bar{q}} e_i^2 \, f^i(x, p_T, \zeta, \mu) \, D^i(z, k_T, \zeta_h, \mu) \, S(l_T, \mu)$$

- no $\int d^2 k_T$ in parton densities $\rightsquigarrow$ no DGLAP type evolution !!

- evolution in rapidity parameters $\zeta$, $\zeta_h$ with $\zeta \zeta_h = Q^2$
  $\rightsquigarrow$ Collins-Soper equation $\rightsquigarrow$ Sudakov factor $\rightsquigarrow$ later slide

- various azimuthal and spin asymmetries
SIDIS at low $q_T$

- Factorization formula

$$\frac{d\sigma_{\gamma^* p}}{dz \, dq_T^2} = (\text{kin. fact.}) \times |H(\mu)|^2 \int d^2 p_T \, d^2 k_T \, d^2 l_T \, \delta^{(2)}(p_T - k_T + l_T + q_T)$$

$$\times \sum_{i=q,\bar{q}} e_i^2 \, f^i(x, p_T, \zeta, \mu) \, D^i(z, k_T, \zeta_h, \mu) \, S(l_T, \mu)$$

- Simplifies if Fourier transform

$$f(p_T) \rightarrow f(b), \quad D(k_T) \rightarrow D(b), \quad S(l_T) \rightarrow S(b):$$

$$\frac{d\sigma_{\gamma^* p}}{dz \, dq_T^2} = (\text{k.f.}) \times |H(\mu)|^2 \int d^2 b \, e^{-ibq_T} \sum_{i=q,\bar{q}} e_i^2 \, f^i(x, b, \zeta, \mu) \, D^i(z, b, \zeta_h, \mu) \, S(b, \mu)$$

Note: $b$ here not the same as $b$ in GPDs (will later call $z$)

- Redefine $f$ and $D$ to each absorb factor $\sqrt{S}$
SIDIS at low $q_T$

- Collins-Soper equation and RGE for $f$ (same for $D$):

$$\frac{d}{d \ln \sqrt{\zeta}} f(x, b, \zeta, \mu) = K(b, \mu) f(x, b, \zeta, \mu)$$

$$\frac{d}{d \ln \mu} f(x, b, \zeta, \mu) = \gamma_F(\zeta, \mu) f(x, b, \zeta, \mu) \quad \text{(no $x$ integral as in DGLAP eq.)}$$

- “cusp anomalous dimension”

$$\frac{dK(b, \mu)}{d \ln \mu} = \frac{d\gamma_F(\zeta, \mu)}{d \ln \sqrt{\zeta}} = -\gamma_K(\mu) = -C_F \frac{2\alpha_s(\mu)}{\pi} + \ldots$$

$$C_F = \frac{N_c^2 - 1}{2N_c}$$
SIDIS at low $q_T$

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$$\frac{d}{d \ln \mu} f(x, b, \zeta, \mu) = \gamma_F(\zeta, \mu) f(x, b, \zeta, \mu) \quad \text{(no $x$ integral as in DGLAP eq.)}$$

- solution:

$$\frac{f(x, b, \zeta, \mu)}{f(x, b, \zeta_0, \mu_0)} = \exp\left\{ - \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \left[ \gamma_K(\mu') \ln \frac{\sqrt{\zeta}}{\mu'} - \gamma_F(\mu'^2, \mu') \right] + K(b, \mu_0) \ln \frac{\sqrt{\zeta}}{\sqrt{\zeta_0}} \right\}$$

- $\exp\{\ldots\} =$ Sudakov factor

- in exponent have “double logarithms” $\ln^2(\mu/\mu_0)$ for $\zeta \sim \mu^2$

- in cross section set $\sqrt{\zeta} \sim \mu \sim Q$ and $\sqrt{\zeta_0} \sim \mu_0 \sim q_T$

- $K(b, \mu)$ calculable in pert. theory only if $b$ is small

- need to interpolate between small and large $b$
Compare with collinear factorization for large $q_T$

\[
\frac{d\sigma_{\gamma^* p}}{dz \, dq_T^2} = (\text{kin. fact.}) \times \int_x^1 \frac{d\hat{x}}{\hat{x}} \int_z^1 \frac{d\hat{z}}{\hat{z}} \delta\left(\frac{q_T^2}{Q^2} - \frac{(1 - \hat{x})(1 - \hat{z})}{\hat{x}\hat{z}}\right) \times \sum_{i,j=q,\bar{q},g} f_i\left(\frac{x}{\hat{x}}, \mu^2\right) D_j\left(\frac{z}{\hat{z}}, \mu^2\right) C_{ij}\left(\hat{x}, \hat{z}, \ln \frac{\mu^2}{Q^2}\right)
\]

- $C_{ij}$ start at $O(\alpha_s)$, must emit partons recoiling against $\vec{q}_T$
- convolution in momentum fractions
Wilson lines in short-distance factorization

- exchange of $>2$ partons between $H$ and $B$ power suppressed
- except for $A^+$ gluon exchange
  $\sim$ resum to all orders

- $H^\mu(l)$ all components big
- $B^\mu(l) \propto p^\mu$ only plus-component big (for right-moving hadron)
  $\sim \ H^\mu \ B^\mu \approx H^- B^+$
Wilson lines in short-distance factorization

- exchange of $> 2$ partons between $H$ and $B$ power suppressed
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  $\sim\rightarrow H_\mu B^\mu \approx H^- B^+$

- Ward identities $\sim\rightarrow$ gluons removed from $H$
in $B$ obtain Wilson line $W(a, b) = P \exp \left[ i g \int_a^b dz^- A^+(z) \right]_{z^+ = 0, \bar{z} = 0}$

$$q(x) \propto \int dz^- e^{i x p^+ z^-} \langle p | \bar{q}(0) W(0, z) \gamma^+ q(z) | p \rangle |_{z^+ = 0, \bar{z} = 0}$$

$\sim\rightarrow$ workshop session
Wilson lines in TMDs

\[ q(x, \vec{k}) \propto \int d\bar{z}^- d^2\vec{z} e^{ixp^+z^-} e^{-i\vec{k}\cdot\vec{z}} \langle p| \bar{q}(0) W_P(0, z) \gamma^+ q(z) |p\rangle \bigg|_{z^+ = 0} \]

- space-time structure of process \( \leadsto \) path \( P \) in Wilson line
- SIDIS: interactions after quark struck by photon
- DY: interactions before quark annihilates

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QCD and hadron structure
Wilson lines in TMDs

\[ q(x, \vec{k}) \propto \int dz^- d^2 \vec{z} e^{ixp^+z^-} e^{-i\vec{k} \cdot \vec{z}} \langle p| \bar{q}(0) W_P(0, z) \gamma^+ q(z) |p \rangle \bigg|_{z^+=0} \]

- space-time structure of process \( \sim \) path \( P \) in Wilson line
- **SIDIS**: interactions after quark struck by photon
- **DY**: interactions before quark annihilates
- obtain “staple like” paths
  - Feynman gauge: pieces at \( z^- \to \pm \infty \) not important
  - light-cone gauge \( A^+ = 0 \): straight sections \( \to 1 \)
    all effects from \( z^- \to \pm \infty \)
Rapidity divergences revisited

- arise as $\int_0^\infty d\ell^+ / \ell^+$ from region $\ell^+ \to 0$ at nonzero $\ell^-$
  $\Rightarrow$ negative rapidity, should not be inside TMD

- need to regulate $\Rightarrow$ rapidity “cutoff” parameter $\zeta$

$$\frac{d}{d \ln \sqrt{\zeta}} = \frac{d}{d(\text{rapidity})}$$

- $\int d^2 k_T$ divergences cancel between real and virtual graphs
  $\Rightarrow$ not present in usual PDFs (or GPDs)
Wilson line has **physical** consequences

- **transverse** proton polarization $\leadsto$ anisotropic $\vec{k}$ distribution

\[
f_{q/p\uparrow}(x, \vec{k}) = f_1(x, \vec{k}^2) + \frac{(\vec{S} \times \vec{k}) \cdot \vec{p}}{m|\vec{p}|} f_{T1\perp}(x, \vec{k}^2)
\]

- induces anisotropic $p_T$ distribution in SIDIS (Sivers effect) observed experimentally

- time reversal changes sign of $(\vec{S} \times \vec{k}) \cdot \vec{p}$

$\leadsto$ Sivers function $= 0$??

Wilson line has **physical** consequences

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- time reversal changes sign of $(\vec{S} \times \vec{k}) \cdot \vec{p}$
  $\rightsquigarrow$ Sivers function $= 0$ ??

- **no**: time reversal interchanges Wilson lines for SIDIS (future pointing) and DY (past pointing)

  \[
  \rightsquigarrow f_{1T,\text{SIDIS}}(x, \vec{k}^2) = -f_{1T,\text{DY}}(x, \vec{k}^2)
  \]

J. Collins '02
Chromodynamic lensing

Sivers effect found in explicit model calculations

S. Brodsky, D.-S. Hwang, I. Schmidt '02

- same model relates anisotropies in
  transv. momentum distribution $f_{q/p\uparrow}(x, \vec{k})$ and
  impact parameter distribution $q_{q/p\uparrow}(x, \vec{b})$

  Burkardt, Hwang '03; Lu, Schmidt '06; Meissner, Metz, Goeke '07

  $q_{q/p\uparrow}(x, \vec{b}) = q(x, \vec{b}^2) - \frac{(\vec{S} \times \vec{b}) \cdot \vec{p}}{m_p |\vec{p}|} \frac{\partial}{\partial \vec{b}^2} e_q(x, \vec{b}^2)$

- anomalous magnetic moments $\kappa_p, \kappa_n$

  $\leadsto$ signs of $\kappa_q = \int dx \int d^2\vec{b} e_q(x, \vec{b}^2)$

  $\leadsto$ signs of Sivers functions for $u$ and $d$ quarks consistent with measurement
More complicated processes

- examples: $pp \rightarrow \gamma + \text{jet} + X$, $pp \rightarrow \pi + \text{jet} + X$

- more partons in initial and final state
  - more complicated Wilson lines
  - more parton densities and fragm. functions

Bomhof, Mulders, Pijlman, Buffing '04-'15

- two-loop analysis $\rightsquigarrow$ breakdown of TMD factorization

Mulders, Rogers '10
Relation between high-$q_T$ and low-$q_T$ descriptions

- for $q_T \gg m$ calc. $k_T$ dependent densities from coll. ones:

\[
f^i(x, k_T^2; \zeta, \mu) = \frac{1}{k_T^2} \sum_j \int_x^1 \frac{dx'}{x'} K^{ij}(x', \ln \frac{k_T^2}{\zeta}) f^j(x'; \mu)
\]

$K$ closely related with DGLAP splitting functions $P$
Relation between high-$q_T$ and low-$q_T$ descriptions

- for $q_T \gg m$ calc. $k_T$ dependent densities from coll. ones:

\[
f_i(x, b_T^2; \zeta, \mu) = f_i^1(x; \mu) + \sum_j \int_x^1 \frac{dx'}{x'} \tilde{K}^{ij} \left( \frac{x}{x'}, \ln \frac{\mu^2}{\zeta}, \ln(\mu^2 b_T^2) \right) f_j^1(x'; \mu)
\]

$\tilde{K}$ closely related with DGLAP splitting functions $P$
Comparison between high-$q_T$ and low-$q_T$ descriptions

- Collinear fact. requires $q_T \gg m$
- $k_T$ fact. requires $q_T \ll Q$
- $\rightsquigarrow$ in region $m \ll q_T \ll Q$ both approaches are valid

- Compare $q_T \gg m$ limit of $k_T$ fact. result
  with $q_T \ll Q$ limit of coll. fact. result
  $\rightsquigarrow$ full agreement for unpol. cross section

  Collins, Soper, Sterman '85; Bacchetta et al. '08

- Detailed comparison also for various spin asymmetries
  e.g. Sivers asy. in SIDIS or Drell-Yan at low $q_T$ (Sivers fct.) and
  high $q_T$ (Qiu-Sterman fct.)

  Ji, Qiu, Vogelsang, Yuan '06; Koike, Vogelsang, Yuan '07
Correspondence at level of graphs

high-$q_T$ calculation

\[ (k-l)^+ \]

low-$q_T$ calculation with $q_T \gg m$

\[ D(z,k_T) \]
Correspondence at level of graphs

high-$q_T$ calculation

low-$q_T$ calculation with $q_T \gg m$
Transverse momentum vs. position

- variables related by 2d Fourier transforms, e.g.
  - quark fields $\tilde{q}(\vec{k}, z^-) = \int d^2 \vec{z} e^{i\vec{z}\vec{k}} q(\vec{z}, z^-)$
  - proton states $|p^+, \vec{b}\rangle = \int d^2 \vec{p} e^{-i\vec{b}\vec{p}} |p^+, \vec{p}\rangle$

- in bilinear operators

$$
\tilde{q}(\vec{k}) \tilde{q}(\vec{l}) = \int d^2 \vec{y} d^2 \vec{z} e^{-i(\vec{y}\vec{k} - \vec{z}\vec{l})} \tilde{q}(\vec{y}) q(\vec{z})
$$

$$
\vec{y}\vec{k} - \vec{z}\vec{l} = \frac{1}{2}(\vec{y} + \vec{z})(\vec{k} - \vec{l}) + \frac{1}{2}(\vec{y} - \vec{z})(\vec{k} + \vec{l})
$$

'average' transv. momentum $\leftrightarrow$ position difference
transv. momentum transfer $\leftrightarrow$ 'average' position

- 'average' transv. mom. and position not Fourier conjugate
Mind the difference

TMDs

\[ \int d^2 \vec{z} e^{-i\vec{z}\vec{k}} \langle \vec{0} \left| \bar{q}(-\frac{1}{2}\vec{z}) \ldots q(\frac{1}{2}\vec{z}) \right| \vec{0} \rangle \]

impact parameter distributions

\[ \int d^2 \Delta e^{-i\vec{b}\Delta} \langle -\frac{1}{2}\Delta | \bar{q}(\vec{0}) \ldots q(\vec{0}) | \frac{1}{2}\Delta \rangle \]

(longitudinal variables not shown for simplicity)

Fourier conjugates:

average transv. momentum \[ q(x, \vec{k}) \] ↔ difference of transv. positions

difference of transv. momenta \[ H(x, \vec{\Delta})_{\xi=0} \] ↔ average transv. position \[ q(x, \vec{b}) \]

Wilson lines, Sudakov resummation, …
Mind the difference

TMDs

\[ \int d^2 \vec{z} e^{-i \vec{z} \cdot \vec{k}} \langle \vec{0} | \bar{q}(-\frac{1}{2} \vec{z}) \ldots q(\frac{1}{2} \vec{z}) | \vec{0} \rangle \]

impact parameter distributions

\[ \int d^2 \Delta e^{-i \vec{b} \cdot \Delta} \langle -\frac{1}{2} \Delta | \bar{q}(\vec{0}) \ldots q(\vec{0}) | \frac{1}{2} \Delta \rangle \]

more general:

GTMDs

\[ \int d^2 \vec{z} e^{-i \vec{z} \cdot \vec{k}} \langle -\frac{1}{2} \Delta | \bar{q}(-\frac{1}{2} \vec{z}) \ldots q(\frac{1}{2} \vec{z}) | \frac{1}{2} \Delta \rangle \]
Mind the difference

TMDs

\[ \int d^2 \vec{z} e^{-i\vec{z}\vec{k}} \langle \vec{0} | \bar{q}(-\frac{1}{2}\vec{z}) \ldots q(\frac{1}{2}\vec{z}) | \vec{0} \rangle \]

impact parameter distributions

\[ \int d^2 \vec{\Delta} e^{-i\vec{b}\vec{\Delta}} \langle -\frac{1}{2}\vec{\Delta} | \bar{q}(\vec{0}) \ldots q(\vec{0}) | \frac{1}{2}\vec{\Delta} \rangle \]

more general:

GTMDs

\[ \int d^2 \vec{z} e^{-i\vec{z}\vec{k}} \langle -\frac{1}{2}\vec{\Delta} | \bar{q}(-\frac{1}{2}\vec{z}) \ldots q(\frac{1}{2}\vec{z}) | \frac{1}{2}\vec{\Delta} \rangle \]

Fourier transf. from \( \vec{\Delta} \) to \( \vec{b} \)

\( \sim \) Wigner functions

parton momentum and position within limits of uncertainty rel’n
### Relations

\[
\begin{align*}
    H(x, z, \Delta) & \quad \leftrightarrow \quad H(x, k, \Delta) \\
    W(x, k, b) & \quad \leftrightarrow \quad W(x, k, b) \\
    q(x, z) & \quad \leftrightarrow \quad q(x, k) \\
    H(x, \Delta) & \quad \leftrightarrow \quad H(x, \Delta) \\
    q(x, \Delta) & \quad \leftrightarrow \quad q(x, b) \\
    F_n(\Delta) & \quad \leftrightarrow \quad F_n(\Delta)
\end{align*}
\]

\[\int d^2 \vec{k} \] needs UV regularization

GPDs taken at zero skewness \( \xi = 0 \)

\( \vec{k} \) dep’t functions have \( \zeta \) dep’ce
More relations

\[ H(k, P, \Delta) \]
\[ H(x, k, \xi, \Delta) \]
\[ H(x, \xi, \Delta^2) \]
\[ \sum_{n}^{\infty} \text{Ak}(\Delta^2)(2\xi)^n \]

\[ f(k, P) \]
\[ f(x, b) \]
\[ f(x, z) \]
\[ f(x, k) \]
\[ f(x) \]

\[ \int d^2 \vec{k} \]
\[ \int d^2 b \]
\[ \int dx x^{n-1} \]

\[ \int dx x^{-1} \]

\[ \Delta = 0 \]
\[ \xi = 0 \]

\[ \text{parton correlation function} \]
\[ \text{Wigner distribution} \]
\[ \text{impact parameter distribution} \]
\[ \text{form factor} \]

\[ \int d^2 \vec{k} \] needs UV regularization
\[ \vec{k} \] dep’t functions have \( \xi \) dep’ce
naive:

\[ \int d^2 \vec{k} q(x, \vec{k}) = q(x) \]

cannot be true because \( q(x, \vec{k}) \sim 1/\vec{k}^2 \) at large \( \vec{k} \)

correct:

\[ \int d^2 \vec{k} q(x, \vec{k}; \zeta, \mu) = q(x; \mu) + \text{calculable terms of } \mathcal{O}(\alpha_s) \]

\( \vec{k}^2 < \mu^2 \)

Fourier trf. from \( \vec{k} \) to \( \vec{z} \):

instead of \( \int d^2 \vec{k} \) have \( \int d^2 \vec{k} e^{i\vec{k}\vec{z}} \) with \( |\vec{z}| = 1/\mu \)

oscillations suppress region \( |\vec{k}| \gg 1/|\vec{z}| \)

\[ q(x, \vec{z}; \zeta, \mu) = q(x; \mu) + \text{calculable terms of } \mathcal{O}(\alpha_s) \]

see earlier slide
Relation with orbital angular momentum

- in GPD lecture have seen two different definitions of orbital angular quark momentum
- construction using Wigner functions:

\[ L^z = \int dx \int d^2 \vec{k} \ d^2 \vec{b} \ (\vec{b} \times \vec{k})^z \ W(x, \vec{k}, \vec{b}) \]

- corresponds to classical intuition
Relation with orbital angular momentum

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- construction using Wigner functions:

\[ L^z = \int dx \int d^2 \vec{k} \ d^2 \vec{b} \ (\vec{b} \times \vec{k})^z \ W(x, \vec{k}, \vec{b}) \]

- corresponds to classical intuition

- but: need Wilson line between fields \( \bar{q}(-\frac{1}{2}z) \) and \( q(\frac{1}{2}z) \) in \( W \)
  - staple-like path \( \sim L^q_{\text{Bashinski, Jaffe}} \)
  - straight-line path \( \sim L^q_{\text{Ji}} \)

  dynamical interpretation of difference:
  \( L \) includes effects of initial/final state interactions  
  
  M Burkardt '13
Summary of part 5

▶ TMD factorisation for measured $p_T \ll$ hard scale
▶ important differences with collinear factorisation, different evolution
▶ subtle dynamical effects due to gluons $\leadsto$ Wilson lines
▶ valid for restricted class of processes
  for some cases smooth theoretical transition to high $p_T$ regime
▶ theoretically controlled access to transverse parton momentum
▶ Wigner functions: unifying framework for describing
  transverse momentum and position