QCD and hadron structure
Part 5: TMDs

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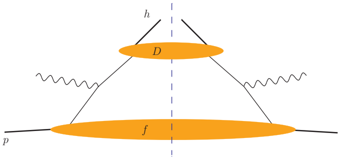
Ecole Joliot Curie 2018

Measured transverse momentum



- consider
  - Drell-Yan with measured small q_T of γ^*
  - SIDIS with measured small q_T of hadron
  - $e^+e^- \to h_1 h_2 + X$ with h_1, h_2 approx. opposite momenta and small relative q_T

- $k_T \sim m$ from collinear graphs matters in final state
  - can still neglect parton k_T in hard scattering
  - but do not $\int d^2k_T$ in parton densities and fragm. fcts.
    $\rightsquigarrow$ k_T dependent/unintegrated PDFs
    also called TMDs (transverse-momentum distributions)

- theoretical framework: TMD factorization
  - also called k_T factorization
    different from (but related to) k_T factorization at small x

k_T dependent parton densities

k_T integrated:

$$f_1(x) = \int \frac{dz^-}{4\pi}\, e^{iz^- p^+ x}\, \langle p,s \big| \bar{q}(0)\gamma^+ W(0,\infty) W(\infty,z^-)\, q(z^-) \big| p,s \rangle \Big|_{z^+=0,\, z_T=0}$$

k_T dependent:

$$\int \frac{dz^-}{4\pi} \frac{d^2 z_T}{(2\pi)^2}\, e^{iz^- p^+ x}\, e^{-i\vec{k}_T \vec{z}_T} \langle p,s \big| \bar{q}(0)\gamma^+ W(0,\infty) W(\infty,z^-,z_T)\, q(z^-,z_T) \big| p,s \rangle \Big|_{z^+=0}$$

- fields at different transv. positions
  implications on Wilson lines $\rightarrow$ later

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$$= f_1(x, k_T^2) - \frac{\epsilon^{ij} k_T^i s_T^j}{m}\, f_{1T}^{\perp}(x, k_T^2)$$

$$\epsilon^{12} = -\epsilon^{21} = 1 \qquad \epsilon^{11} = \epsilon^{22} = 0$$

- fields at different transv. positions
  implications on Wilson lines $\rightarrow$ later
- correlations between spins and transv. momentum
  e.g. Sivers function $f_{1T}^{\perp}$

A zoo of distributions

- collinear twist 2 densities:

  f_1 unpol. quark in unpol. proton
  g_1 correlate s_L of quark with S_L of proton
  h_1 correlate s_T of quark with S_T of proton

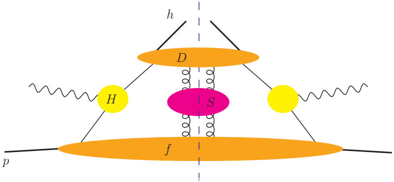
- k_T dependent twist 2 densities:

  f_1, g_1, h_1 as above
  $f_{1T}^{\perp}$ correlate k_T of quark with S_T of proton (Sivers)
  $h_1^{\perp}$ correlate k_T and s_T of quark (Boer-Mulders)
  $g_{1T}, h_{1T}^{\perp}, h_{1L}^{\perp}$ three more densities

- analogous for fragmentation functions:
  - $f_1 \leftrightarrow D_1$ unpolarized
  - $h_1^{\perp} \leftrightarrow H_1^{\perp}$ Collins fragm. fct.

TMD factorization (SIDIS as example)

- take Q large and q_T small ($\sim m$ for power counting purposes)



- transverse-momentum dep't distribution and fragmentation fcts.
- only virtual corrections to hard subgraph
  no radiation of high-p_T partons allowed

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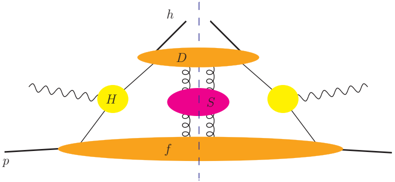
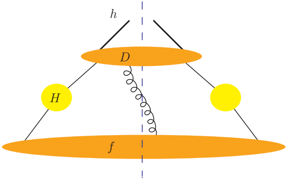


- soft gluon exchange does not cancel in sum over hadronic final state at leading-power accuracy gives soft factor in factorization formula

$S =$ universal non-perturbative fct
$\rightarrow 1$ when integrate over k_T
cancellation between real



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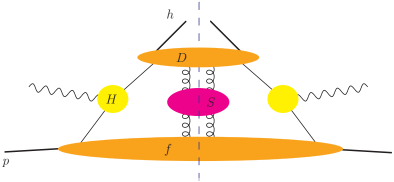

- soft gluon exchange does not cancel in sum over hadronic final state at leading-power accuracy gives soft factor in factorization formula

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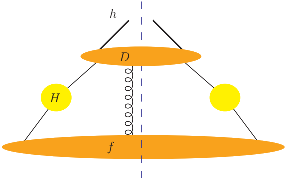
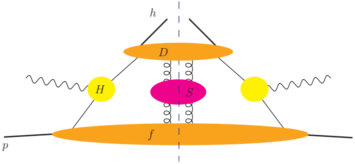


SIDIS at low q_T



- factorization formula

$$\frac{d\sigma_{\gamma^* p}}{dz\, dq_T^2} = (\text{kin. fact.}) \times |H(\mu)|^2 \int d^2p_T\, d^2k_T\, d^2l_T\, \delta^{(2)}\big(\vec{p}_T - \vec{k}_T + \vec{l}_T + \vec{q}_T\big) \times \sum_{i=q,\bar{q}} e_i^2\, f^i(x, p_T, \zeta, \mu)\, D^i(z, k_T, \zeta_h, \mu)\, S(l_T, \mu)$$

- no $\int d^2k_T$ in parton densities $\rightsquigarrow$ no DGLAP type evolution !!
- evolution in rapidity parameters ζ, ζ_h with $\zeta\zeta_h = Q^2$
  $\rightsquigarrow$ Collins-Soper equation $\rightsquigarrow$ Sudakov factor $\rightsquigarrow$ later slide
- various azimuthal and spin asymmetries

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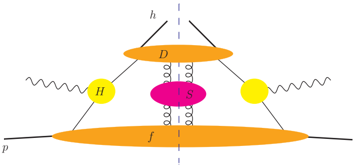

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- simplifies if Fourier transform
  $f(p_T) \to f(b)$, $D(k_T) \to D(b)$, $S(l_T) \to S(b)$:

$$\frac{d\sigma_{\gamma^* p}}{dz\, dq_T^2} = (\text{k.f.}) \times |H(\mu)|^2 \int d^2b\, e^{-i\vec{b}\vec{q}_T} \sum_{i=q,\bar{q}} e_i^2\, f^i(x, b, \zeta, \mu)\, D^i(z, b, \zeta_h, \mu)\, S(b, \mu)$$

  note: b here not the same as b in GPDs (will later call z)

- redefine f and D to each absorb factor $\sqrt{S}$

SIDIS at low q_T

- Collins-Soper equation and RGE for f (same for D):

$$\frac{d}{d\ln\sqrt{\zeta}}\, f(x,b,\zeta,\mu) = K(b,\mu)\, f(x,b,\zeta,\mu)$$

$$\frac{d}{d\ln\mu}\, f(x,b,\zeta,\mu) = \gamma_F(\zeta,\mu)\, f(x,b,\zeta,\mu)$$

(no x integral as in DGLAP eq.)

- "cusp anomalous dimension"

$$\frac{dK(b,\mu)}{d\ln\mu} = \frac{d\gamma_F(\zeta,\mu)}{d\ln\sqrt{\zeta}} = -\gamma_K(\mu) = -C_F\,\frac{2\alpha_s(\mu)}{\pi} + \ldots$$

$C_F = \frac{N_c^2-1}{2N_c}$

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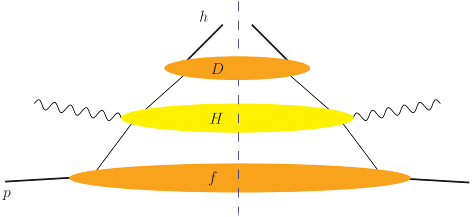
$$\frac{d}{d\ln\mu}\, f(x,b,\zeta,\mu) = \gamma_F(\zeta,\mu)\, f(x,b,\zeta,\mu) \quad \text{(no } x \text{ integral as in DGLAP eq.)}$$

- solution:

$$\frac{f(x,b,\zeta,\mu)}{f(x,b,\zeta_0,\mu_0)} = \exp\Bigg\{-\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\left[\gamma_K(\mu')\ln\frac{\sqrt{\zeta}}{\mu'} - \gamma_F({\mu'}^2,\mu')\right] + K(b,\mu_0)\ln\frac{\sqrt{\zeta}}{\sqrt{\zeta_0}}\Bigg\}$$

- $\exp\{\ldots\}$ = Sudakov factor
- in exponent have "double logarithms" $\ln^2(\mu/\mu_0)$ for $\zeta \sim \mu^2$
  in cross section set $\sqrt{\zeta} \sim \mu \sim Q$ and $\sqrt{\zeta_0} \sim \mu_0 \sim q_T$
- $K(b,\mu)$ calculable in pert. theory only if b is small
  need to interpolate between small and large b

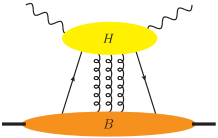
Compare with collinear factorization for large q_T



$$\frac{d\sigma_{\gamma^* p}}{dz\, dq_T^2} = (\text{kin. fact.}) \times \int_x^1 \frac{d\hat{x}}{\hat{x}} \int_z^1 \frac{d\hat{z}}{\hat{z}}\, \delta\left(\frac{{q_T}^2}{Q^2} - \frac{(1-\hat{x})(1-\hat{z})}{\hat{x}\hat{z}} \right) \times \sum_{i,j=q,\bar{q},g} f_i\left(\frac{x}{\hat{x}}, \mu^2\right) D_j\left(\frac{z}{\hat{z}}, \mu^2\right) C_{ij}\left(\hat{x}, \hat{z}, \ln\frac{\mu^2}{Q^2}\right)$$

- C_{ij} start at $\mathcal{O}(\alpha_s)$, must emit partons recoiling against $\vec{q}_T$
- convolution in momentum fractions

Wilson lines in short-distance factorization



- exchange of > 2 partons between H and B power suppressed
- except for A^+ gluon exchange
  $\rightsquigarrow$ resum to all orders

- $H^\mu(l)$ all components big
  $B^\mu(l) \propto p^\mu$ only plus-component big (for right-moving hadron)
  $\rightsquigarrow \quad H_\mu \, B^\mu \approx H^- B^+$

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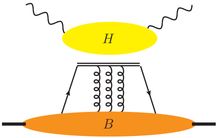

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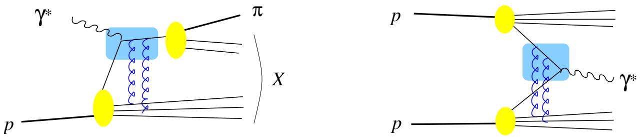
- Ward identities $\rightsquigarrow$ gluons removed from H
  in B obtain Wilson line $W(a,b) = \mathrm{P} \exp \Big[ig \int_a^b dz^- \, A^+(z) \Big]_{z^+=0,\, \vec{z}=\vec{0}}$

$$q(x) \propto \int dz^- \, e^{ixp^+z^-} \langle p | \, \bar{q}(0) \, W(0,z) \, \gamma^+ q(z) \, | p \rangle \Big|_{z^+=0,\, \vec{z}=\vec{0}}$$

$\rightsquigarrow$ workshop session

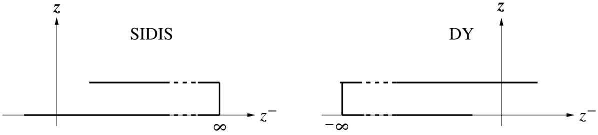
Wilson lines in TMDs



$$q(x,\vec{k}) \propto \int dz^{-}\, d^2\vec{z}\, e^{ixp^{+}z^{-}}\, e^{-i\vec{k}\,\vec{z}} \langle p|\, \bar{q}(0)\, W_P(0,z)\, \gamma^{+} q(z)\, |p\rangle \Big|_{z^{+}=0}$$

- space-time structure of process $\rightsquigarrow$ path P in Wilson line
- SIDIS: interactions after quark struck by photon
  DY: interactions before quark annihilates

Wilson lines in TMDs



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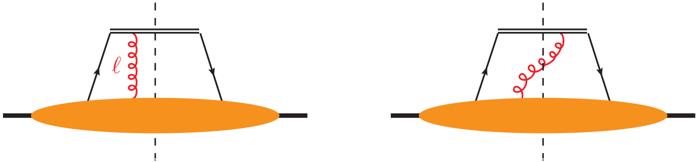
- space-time structure of process $\rightsquigarrow$ path P in Wilson line
- SIDIS: interactions after quark struck by photon
  DY: interactions before quark annihilates
- obtain "staple like" paths
  - Feynman gauge: pieces at $z^- \to \pm\infty$ not important
  - light-cone gauge $A^+ = 0$: straight sections $\to 1$
    all effects from $z^- \to \pm\infty$

Rapidity divergences revisited

- arise as $\int_0 d\ell^+/\ell^+$ from region $\ell^+ \to 0$ at nonzero ℓ^-
  $\rightsquigarrow$ negative rapidity, should not be inside TMD
- need to regulate $\rightsquigarrow$ rapidity "cutoff" parameter ζ

$$\frac{d}{d\ln\sqrt{\zeta}} = \frac{d}{d(\text{rapidity})}$$



- $\int d^2k_T$ divergences cancel between real and virtual graphs
  $\rightsquigarrow$ not present in usual PDFs (or GPDs)

Wilson line has physical consequences

- transverse proton polarization $\rightsquigarrow$ anisotropic $\vec{k}$ distribution

$$f_{q/p\uparrow}(x, \vec{k}) = f_1(x, \vec{k}^2) + \frac{(\vec{S} \times \vec{k}\,) \cdot \vec{p}}{m\,|\vec{p}\,|}\, f_{T1}{\perp}(x, \vec{k}^2)$$

- induces anisotropic p_T distribution in SIDIS (Sivers effect) observed experimentally
- time reversal changes sign of $(\vec{S} \times \vec{k}) \cdot \vec{p}$
  $\rightsquigarrow$ Sivers function $= 0$??

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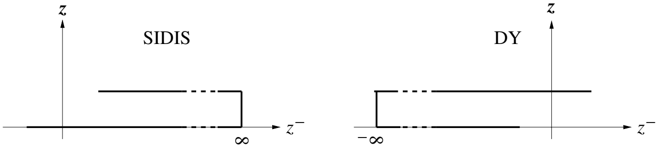
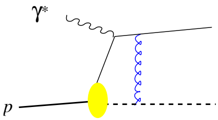
- time reversal changes sign of $(\vec{S}\times\vec{k})\cdot\vec{p}$
  $\rightsquigarrow$ Sivers function $= 0$??
- no: time reversal interchanges Wilson lines for
  SIDIS (future pointing) and DY (past pointing)

$$\rightsquigarrow \quad f_{1T}^{\perp,\mathrm{SIDIS}}(x,\vec{k}^2) = -f_{1T}^{\perp,\mathrm{DY}}(x,\vec{k}^2)$$

J. Collins '02



Chromodynamic lensing



Sivers effect found in explicit model calculations

S. Brodsky, D.-S. Hwang, I. Schmidt '02

- same model relates anisotropies in
  transv. momentum distribution $f_{q/p\uparrow}(x,\vec{k})$ and
  impact parameter distribution $q_{q/p\uparrow}(x,\vec{b})$

  Burkardt, Hwang '03; Lu, Schmidt '06; Meissner, Metz, Goeke '07

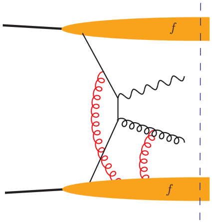
$$q_{q/p\uparrow}(x,\vec{b}) = q(x,\vec{b}^2) - \frac{(\vec{S}\times\vec{b}\,)\cdot\vec{p}}{m_p\,|\vec{p}|}\,\frac{\partial}{\partial\vec{b}^2}e_q(x,\vec{b}^2)$$

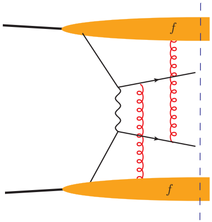
- anomalous magnetic moments κ_p, κ_n
  - ⇝ signs of $\kappa_q = \int dx \int d^2\vec{b}\, e_q(x,\vec{b}^2)$
  - ⇝ signs of Sivers functions for u and d quarks consistent with measurement

More complicated processes

- examples: $pp \to \gamma + \text{jet} + X$, $pp \to \pi + \text{jet} + X$
- more partons in initial and final state
  $\rightsquigarrow$ more complicated Wilson lines
  $\rightsquigarrow$ more parton densities and fragm. functions

Bomhof, Mulders, Pijlman, Buffing '04-'15





- two-loop analysis $\rightsquigarrow$ breakdown of TMD factorization

Mulders, Rogers '10

Relation between high-q_T and low-q_T descriptions

- for $q_T \gg m$ calc. k_T dependent densities from coll. ones:

$$f_1^i(x, k_T^2; \zeta, \mu) = \frac{1}{k_T^2} \sum_j \int_x^1 \frac{dx'}{x'} K^{ij}\Big(\frac{x}{x'}, \ln\frac{k_T^2}{\zeta}\Big) f_1^j(x'; \mu)$$

K closely related with DGLAP splitting functions P

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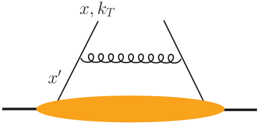
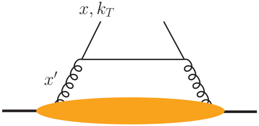




$$f_1^i(x, b_T^2; \zeta, \mu) = f_1^i(x; \mu) + \sum_j \int_x^1 \frac{dx'}{x'}\, \tilde{K}^{ij}\Big(\frac{x}{x'}, \ln\frac{\mu^2}{\zeta}, \ln(\mu^2 b_T^2)\Big)\, f_1^j(x'; \mu)$$

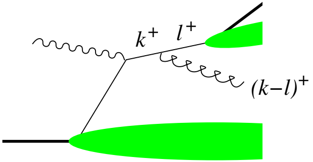
$\tilde{K}$ closely related with DGLAP splitting functions P

Comparison between high-q_T and low-q_T descriptions

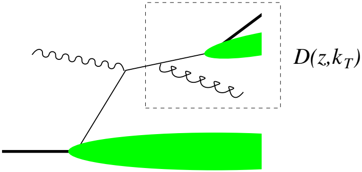
- collinear fact. requires $q_T \gg m$
  k_T fact. requires $q_T \ll Q$
  $\rightsquigarrow$ in region $m \ll q_T \ll Q$ both approaches are valid
- compare $q_T \gg m$ limit of k_T fact. result
  with $q_T \ll Q$ limit of coll. fact. result
  $\rightsquigarrow$ full agreement for unpol. cross section
  Collins, Soper, Sterman '85; Bacchetta et al. '08
- detailed comparison also for various spin asymmetries
  e.g. Sivers asy. in SIDIS or Drell-Yan at low q_T (Sivers fct.) and high q_T (Qiu-Sterman fct.)
  Ji, Qiu, Vogelsang, Yuan '06; Koike, Vogelsang, Yuan '07

Correspondence at level of graphs

high-q_T calculation



low-q_T calculation with $q_T \gg m$



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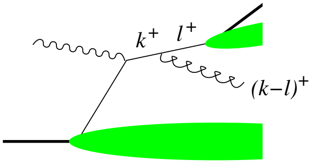

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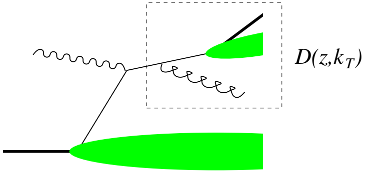
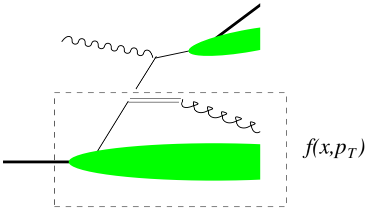




Transverse momentum vs. position

- variables related by 2d Fourier transforms, e.g.
  - quark fields $\tilde{q}(\vec{k}, z^-) = \int d^2\vec{z}\, e^{i\vec{z}\vec{k}} q(\vec{z}, z^-)$
  - proton states $|p^+, \vec{b}\rangle = \int d^2\vec{p}\, e^{-i\vec{b}\vec{p}} |p^+, \vec{p}\rangle$
- in bilinear operators

$$\bar{\tilde{q}}(\vec{k})\, \tilde{q}(\vec{l}) = \int d^2\vec{y}\, d^2\vec{z}\, e^{-i(\vec{y}\vec{k} - \vec{z}\vec{l})}\, \bar{q}(\vec{y})\, q(\vec{z})$$

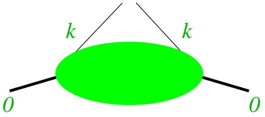
$$\vec{y}\vec{k} - \vec{z}\vec{l} = \tfrac{1}{2}(\vec{y} + \vec{z})(\vec{k} - \vec{l}) + \tfrac{1}{2}(\vec{y} - \vec{z})(\vec{k} + \vec{l})$$

'average' transv. momentum $\leftrightarrow$ position difference
transv. momentum transfer $\leftrightarrow$ 'average' position

- 'average' transv. mom. and position not Fourier conjugate

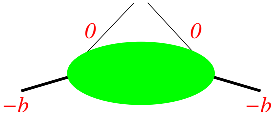
Mind the difference

TMDs



$$\int d^2\vec{z}\, e^{-i\vec{z}\vec{k}}\, \langle\vec{0}|\bar{q}(-\tfrac{1}{2}\vec{z})\dots q(\tfrac{1}{2}\vec{z})|\vec{0}\rangle$$

impact parameter distributions



$$\int d^2\vec{\Delta}\, e^{-i\vec{b}\vec{\Delta}}\, \langle -\tfrac{1}{2}\vec{\Delta}|\bar{q}(\vec{0})\dots q(\vec{0})|\tfrac{1}{2}\vec{\Delta}\rangle$$

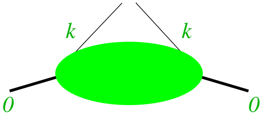
(longitudinal variables not shown for simplicity)

Fourier conjugates:

average transv. momentum	$\leftrightarrow$	difference of transv. positions
$q(x,\vec{k})$		Wilson lines, Sudakov resummation, ...
difference of transv. momenta	$\leftrightarrow$	average transv. position
$H(x,\vec{\Delta})_{\xi=0}$		$q(x,\vec{b})$

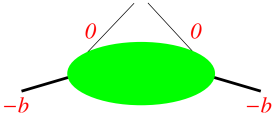
Mind the difference

TMDs



$$\int d^2\vec z\, e^{-i\vec z\vec k}\,\langle\vec 0|\bar q(-\tfrac{1}{2}\vec z)\dots q(\tfrac{1}{2}\vec z)|\vec 0\rangle$$

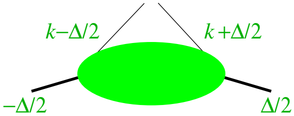
impact parameter distributions



$$\int d^2\vec\Delta\, e^{-i\vec b\vec\Delta}\,\langle -\tfrac{1}{2}\vec\Delta|\bar q(\vec 0)\dots q(\vec 0)|\tfrac{1}{2}\vec\Delta\rangle$$

more general:

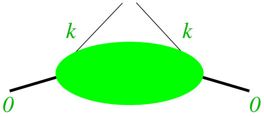
GTMDs



$$\int d^2\vec z\, e^{-i\vec z\vec k}\,\langle -\tfrac{1}{2}\vec\Delta|\bar q(-\tfrac{1}{2}\vec z)\dots q(\tfrac{1}{2}\vec z)|\tfrac{1}{2}\vec\Delta\rangle$$

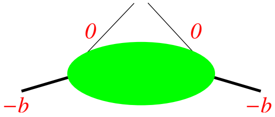
Mind the difference

TMDs



$$\int d^2\vec{z}\, e^{-i\vec{z}\vec{k}}\, \langle \vec{0}|\bar{q}(-\tfrac{1}{2}\vec{z}) \dots q(\tfrac{1}{2}\vec{z})|\vec{0}\rangle$$

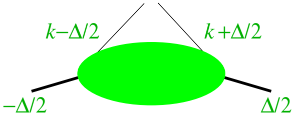
impact parameter distributions



$$\int d^2\vec{\Delta}\, e^{-i\vec{b}\vec{\Delta}}\, \langle -\tfrac{1}{2}\vec{\Delta}|\bar{q}(\vec{0}) \dots q(\vec{0})|\tfrac{1}{2}\vec{\Delta}\rangle$$

more general:

GTMDs



$$\int d^2\vec{z}\, e^{-i\vec{z}\vec{k}}\, \langle -\tfrac{1}{2}\vec{\Delta}|\bar{q}(-\tfrac{1}{2}\vec{z}) \dots q(\tfrac{1}{2}\vec{z})|\tfrac{1}{2}\vec{\Delta}\rangle$$

Fourier transf. from $\vec{\Delta}$ to $\vec{b}$
$\rightsquigarrow$ Wigner functions

parton momentum and position
within limits of uncertainty rel'n

Relations

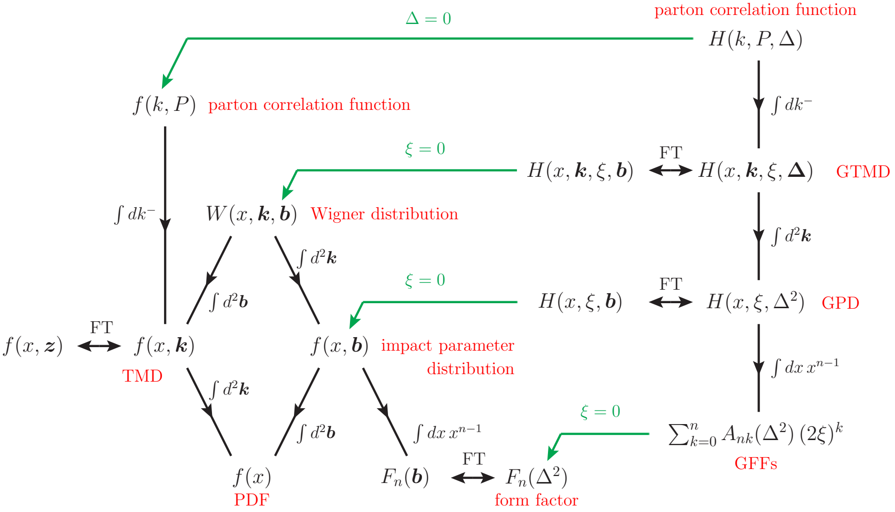
$$
\begin{tikzcd}
H(x,\boldsymbol{z},\boldsymbol{\Delta}) \arrow[r, leftrightarrow, "\mathrm{FT}\ \boldsymbol{z} \leftrightarrow \boldsymbol{k}"'] & H(x,\boldsymbol{k},\boldsymbol{\Delta}) \arrow[r, leftrightarrow, "\mathrm{FT}\ \boldsymbol{\Delta} \leftrightarrow \boldsymbol{b}"] \arrow[d, "\boldsymbol{\Delta} = \mathbf{0}"'] \arrow[dr, "\int d^2\boldsymbol{k}"] & W(x,\boldsymbol{k},\boldsymbol{b}) \arrow[dl, "\int d^2\boldsymbol{b}"] \arrow[dr, "\int d^2\boldsymbol{k}"] & \\
q(x,\boldsymbol{z}) \arrow[r, leftrightarrow, "\mathrm{FT}\ \boldsymbol{z} \leftrightarrow \boldsymbol{k}"'] & q(x,\boldsymbol{k}) \arrow[dr, "\int d^2\boldsymbol{k}"] & H(x,\boldsymbol{\Delta}) \arrow[r, leftrightarrow, "\mathrm{FT}\ \boldsymbol{\Delta} \leftrightarrow \boldsymbol{b}"'] \arrow[d, "\boldsymbol{\Delta} = \mathbf{0}"'] \arrow[dr, "\int dx\, x^{n-1}"] & q(x,\boldsymbol{b}) \arrow[dl, "\int d^2\boldsymbol{b}"'] \\
 & & q(x) & F_n(\boldsymbol{\Delta})
\end{tikzcd}
$$

GPDs taken at zero skewness $\xi = 0$

$\int d^2\vec{k}$ needs UV regularization

$\vec{k}$ dep't functions have ζ dep'ce

More relations



$\int d^2\vec{k}$ needs UV regularization

$\vec{k}$ dep't functions have ζ dep'ce

- naive:

$$\int d^2\vec{k}\, q(x,\vec{k}) = q(x)$$

cannot be true because $q(x,\vec{k}) \sim 1/\vec{k}^2$ at large $\vec{k}$

- correct:

$$\int_{\vec{k}^2<\mu^2} d^2\vec{k}\, q(x,\vec{k};\zeta,\mu) = q(x;\mu) + \text{calculable terms of } \mathcal{O}(\alpha_s)$$

- Fourier trf. from $\vec{k}$ to $\vec{z}$:
  instead of $\int_{\vec{k}^2<\mu^2} d^2\vec{k}$ have $\int d^2\vec{k}\, e^{i\vec{k}\vec{z}}$ with $|\vec{z}| = 1/\mu$

oscillations suppress region $|\vec{k}| \gg 1/|\vec{z}|$

$$q(x,\vec{z};\zeta,\mu) = q(x;\mu) + \text{calculable terms of } \mathcal{O}(\alpha_s)$$

see earlier slide

Relation with orbital angular momentum

- in GPD lecture have seen two different definitions of orbital angular quark momentum
- construction using Wigner functions:

$$L^z = \int dx \int d^2\vec{k}\, d^2\vec{b}\, (\vec{b} \times \vec{k})^z\, W(x, \vec{k}, \vec{b})$$

- corresponds to classical intuition

Relation with orbital angular momentum

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$$L^z = \int dx \int d^2\vec{k}\; d^2\vec{b}\; (\vec{b} \times \vec{k})^z \; W(x, \vec{k}, \vec{b})$$

- corresponds to classical intuition
- but: need Wilson line between fields $\bar{q}\big(-\frac{1}{2}z\big)$ and $q\big(\frac{1}{2}z\big)$ in W
  - staple-like path $\rightsquigarrow \mathcal{L}^q_{\text{Bashinski, Jaffe}}$
  - straight-line path $\rightsquigarrow L^q_{\text{Ji}}$

  dynamical interpretation of difference:
  $\mathcal{L}$ includes effects of initial/final state interactions M Burkardt '13

Summary of part 5

- TMD factorisation for measured $p_T \ll$ hard scale
- important differences with collinear factorisation, different evolution
- subtle dynamical effects due to gluons $\rightsquigarrow$ Wilson lines
- valid for restricted class of processes
  for some cases smooth theoretical transition to high p_T regime
- theoreticallly controlled access to transverse parton momentum
- Wigner functions: unifying framework for describing
  transverse momentum and position